November 10, 2024 Math 8

Handout 10: Logic: Implications and proofs.

Implication and equivalence

In addition to all previous logic operations, there are also operations expressing logical relationships, which we have not yet fully discussed. One is implication, also known as conditional and denoted by $A \Rightarrow B$ (reads A implies B, or "If A, then B"). It is defined by the following truth table:

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

Note that, in all situations where A is false $A \Rightarrow B$, is automatically true. E.g., a statement "if $2 \times 2 = 5$, then. . . " is automatically true, no matter what proposition one puts in place of dots. Every statement (be it true or false) "is implied by"/"follows from" any false statement! One may think of this in terms of a vacuous truth: Every statement is true in all cases when the antecedent false statement is true. "A implies B" indeed only means "whenever A, then B". Another logic operation, which we have already used, is called equivalence and defined as $A \Leftrightarrow B$, which is true if A, B always have the same value (both true or both false).

Exercise 1. Using truth tables, show that $A \Leftrightarrow B$ is equivalent to $A \Rightarrow B$ AND $B \Rightarrow A$.

Logic: proofs

We are commonly asked to prove something: given a series of statements (assumptions), prove another statement (conclusion).

In the simple case where all the statements are just formulas involving some logic variables and basic operations, one way to do it is by writing a truth table: list all possible combinations of values of variables and verify that in each case where all assumptions are true, the conclusion is also true. However, usually that's not how it is done. Instead, we construct a series of intermediate statements, each of which follows from the previous ones and assumptions.

When constructing these intermediate statements, we can again use truth tables or common logic laws:

- Given $A \Rightarrow B$ AND A, we can conclude B (Modus Ponens)
- Given $A \Rightarrow B$ AND $B \Rightarrow C$, we can conclude that $A \Rightarrow C$. [Note: it doesn't mean that in this situation, C is always true! It only means that if A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ AND $\neg B$, we can conclude A
- Given $A \Rightarrow B$ AND $\neg B$, we can conclude $\neg A$ (Modus Tollens)

- $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$ (De Morgan Law)
- $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ (Law of contrapositive)
- [Proof by cases] Given $A \vee B$, $A \Rightarrow C$ AND $B \Rightarrow C$, we can conclude C

Note: it is important to realize that statements $A \Rightarrow B$ and $B \Rightarrow A$ are not equivalent! (They are called converse of each other).

Common methods of proof

Proof by cases.

Example: Prove that for any integer n, the number $n(n+1)$ is even.

Proof. If n is integer, it is either even or odd. If n is even, then $n(n+1)$ is even (a multiple of even is always even). If n is odd, then $n+1$ is even and thus $n(n+1)$ is even too. Thus, in all cases $n(n+1)$ is even. ⊡

General scheme: Given $A_1 \vee A_2$, from $A_1 \Rightarrow B$, AND $A_2 \Rightarrow B$ we can conclude that B is true.

Considering truth tables can be an example of proof by cases. There can be more than two cases. Note: it is important to verify that the cases you consider cover all possibilities (i.e. that at least one of the statements A_1, A_2 is always true). If there are many cases, $1, 2, \ \dots \ , \ n$, we may consider a general case, k, which, depending on the value of k can correspond to any of the possible cases.

Exercise 1: Prove that for any integer n and $k < n$, product $n(n-1) \dots (n-k+1)$ is divisible by k.

Exercise 2: Prove that for any integer n and k, product $n(n+1) \dots (n+k-1)$ is divisible by k.

Exercise 3: Prove that for any integer n and $k < n$, products $n(n-1) \dots (n-k+1)$ and $n(n+1) \dots (n+k-1)$ are divisible by $k!$

Conditional proof.

Example: Prove that if n is even, then n^2 is even.

Proof. Assume that n is even. Then $n^2 = n \cdot n$ is also even, since a multiple of even is even. ⊡

General scheme: To prove $A \Rightarrow B$, we can

- Assume A
- Give a proof of B (in the proof, we can use that A is true).

This proves $A \Rightarrow B$ (without any assumptions).

Exercise 4: Prove that an integer $a = a_1 a_2 \dots a_n$ is divisible by 3 if the sum of its digits, $a_1 + a_2 + \dots + a_n$, is divisible by 3.

Proof by contradiction.

Example: Prove that if x is a real root of polynomial $p(x) = 10x^3 + 2x + 15$, then x must be negative.

Proof. Assume that x is not negative, i.e. $x < 0$. Then $p(x) = 10x^3 + 2x + 15 > 15$, which contradicts the fact that x is a root of $p(x)$. Thus, our assumption can not be true, so x must be negative. ⊡

General scheme: To prove that A is true, assume A is false, and derive a contradiction. This proves that A must be true.

Exercise 5: Prove that for any rational numbers p and q, $q < p$, there exists a rational number, r, in-between, $q < r < p$.

Homework problems

In problems 1, 2, you need to (a) write the obvious conclusion from given statements; and (b) justify the conclusion, by writing a chain of arguments which leads to it. It may help to write the given statements and conclusion by logical formulas (denoting the statements which are used by letters $A, B, \dots$ connected by logical operations $\vee,\ \wedge, \Rightarrow$, ….

1. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore, … .
2. If Jack comes home late from school, it means he either had a track meet or a theater club. After a track meet, he comes home very tired. Today he came home late but was not tired. Therefore, … .
3. Consider the following statement:

   You can't be happy unless you have a clear conscience.

   Can you rewrite it using the usual logic operations such as by logical operations $\vee,\ \wedge, \Rightarrow$,? Use letter H for "you are happy" and C for "you have a clear conscience".
4. You probably know Lewis Carroll as the author of Alice in Wonderland and other books. What you might not know is that he was also a mathematician very much interested in logic and had invented a number of logic puzzles. Here is one of them. You are given 3 statements:
   a. All babies are illogical.
   b. Nobody is despised who can manage a crocodile.
   c. Illogical persons are despised.

   Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic? It might help to write each of them as combination of elementary statements about a given person, e.g. B for "this person is a baby", I for "this person is illogical", etc.
5. Here is another of Lewis Carrol's puzzles.
   a. All hummingbirds are richly colored.
   b. No large birds live on honey.
   c. Birds that do not live on honey are dull in color.

Therefore, . . . (You may assume that "dull in color" is the same as "not richly colored"). Hint: think of all these as statements about some bird X and rewrite in simpler form, using only basic logic operations. E.g., first statement can be rewritten as "If X is a hummingbird, then X is richly colored".

6. Prove by contradiction that there does not exist a smallest positive rational number.
7. Use proof by contradiction to prove the following statement:

   If the square of an integer number n is even, then n itself is even.

You can use without proof that every integer number is either even (i.e., can be written in the form $n = 2k$, with integer k) or odd (i.e., can be written as $n = 2k + 1$, with integer k).

8. Let A, B, C be logical variables. Given that the following are true:

   $(A \vee B)$ AND $(B \Rightarrow \neg C)$ AND $(C \Rightarrow (\neg A \vee B))$

prove that C is false.

Recap: Logic operations

- NOT (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$, $\sim A$, or (in computer science) $!A$.
- OR (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called "inclusive or" to distinguish it from the "exclusive or" described in problem 4 below. Commonly denoted by $A \vee B$.
- AND (for example A AND B): true if both A, B are true, and false otherwise (i.e., if at least one of them is false). Commonly denoted by $A \wedge B$.
- NOR, NOT OR, for example NOT (A OR B), true if both A and B are false, and false if either A or B is true:
  - A NOR $B \Leftrightarrow \neg(A \vee B))$
- NAND, NOT AND, for example NOT (A AND B), true if at least one of A, B is false, and false otherwise (i.e., if both A, B are true):
  - A NAND $B \Leftrightarrow \neg(A \wedge B))$:
- XOR, exclusive OR gate: true if and only if exactly one of A, B is true and false otherwise:
  - A XOR $B \Leftrightarrow (\neg A \wedge B) \vee (A \wedge \neg B))$
- XNOR, equivalence gate, NOT exclusive OR: true if and only if both A, B are either true or false, and false otherwise, if A and B are different:
  - A XNOR $B \Leftrightarrow (\neg A \vee B) \wedge (A \vee \neg B))$

Recap: selected Logic Laws

- Double negation:

$$\neg(\neg A) \Leftrightarrow A$$

- Idempotency:

$$A \vee A \Leftrightarrow A$$

$$A \wedge A \Leftrightarrow A$$

- De Morgan (disjunction and conjunction negation):

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

- Distributive:

$$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$$