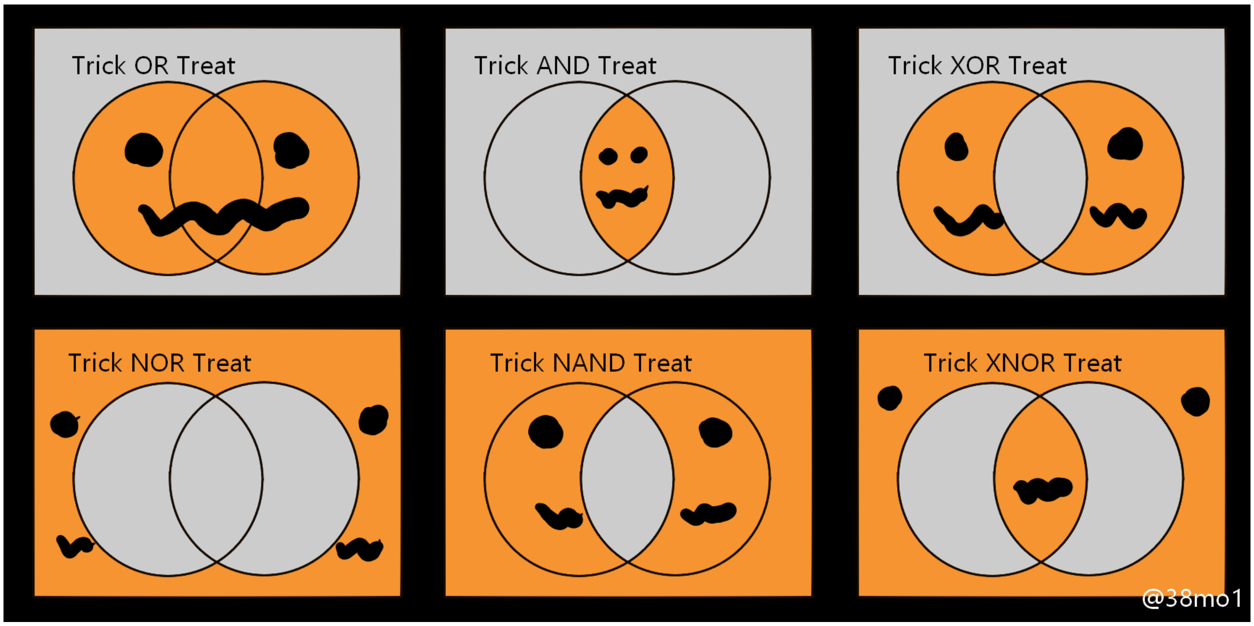
October 20, 2024 Math 8

Handout 6: Mathematical logic

Today we will start discussing formal rules of logic. In logic, we will be dealing with **Boolean** expressions, i.e. expressions which only take two values, TRUE and FALSE. We will commonly use abbreviations T and F for these values. You can also think of these two values as the two possible digits in binary (base 2) arithmetic: $T = 1, F = 0$.

In the usual arithmetic, we have some operations (addition, multiplication, . . .) which satisfy certain laws (associativity, distributivity, ...). Similarly, there are logic operations and logic laws.



Propositions

Proposition is a sentence (statement) which can be evaluated to be either true or false, but not both. If we know that the proposition A is true, we can write $V(A) = T$ to denote that valuation of the proposition A is TRUE. Otherwise, $V(A) = F$.

Simple sentences, which are true, or false, are basic propositions. Larger and more complex sentences can be constructed from basic propositions by combining them with the logic operations.

Basic logic operations

- NOT (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$, $\sim A$, or (in computer science) $!A$.
- AND (for example A AND B): true if both A, B are true, and false otherwise (i.e., if at least one of them is false). Commonly denoted by $A \land B$.
- OR (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called "inclusive or" to distinguish it from the "exclusive or" described in problem 4 below. Commonly denoted by $A \lor B$.

Truth tables

If we have a logical formula involving variables $A, B, C, \ldots$, we can make a table listing, for every possible combination of values of $A, B, \ldots$, the value of our formula. For example, the following is the truth tables for OR and AND:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	A AND B
T	T	T
T	F	F
F	T	F
F	F	F

Logic Laws

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example, $A \vee B$ is the same as $B \vee A$. Here, "the same" means "for all values of A, B, these two expressions give the same answer"; it is usually denoted by the equivalence sign, $A \Leftrightarrow B$. Using this the laws of the logic operations can be written as

- Commutative:

$$A \vee B \Leftrightarrow B \vee A$$

$$A \wedge B \Leftrightarrow B \wedge A$$

- Associative:

$$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$$

$$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$$

- Distributive:

$$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$$

$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$$

- De Morgan (negation):

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

- Idempotency:

$$A \vee A \Leftrightarrow A$$

$$A \wedge A \Leftrightarrow A$$

- Excluded middle:

$$A \vee \neg A \Leftrightarrow T$$

$$A \wedge \neg A \Leftrightarrow F$$

- Identity:

$$A \vee F \Leftrightarrow A$$

$$A \wedge T \Leftrightarrow A$$

- Domination:

$$A \vee T \Leftrightarrow T$$

$$A \wedge F \Leftrightarrow F$$

Truth tables provide the most straightforward (but not the shortest) way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

Exercise 1. Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?

- $(A \vee B) \wedge (A \vee C)$
- $A \vee (B \wedge C)$

Exercise 2. Using truth tables show that commutative and associative laws hold for $\vee$ and $\wedge$.

Exercise 3. Using truth tables prove de Morgan's negation laws.

Summary of logic operations

NOT negation	AND conjunction	OR disjunction	IF_THEN sufficient	ONLY_IF necessary	IF_AND_ONLY_IF equivalent
$\neg, \sim, \bar{\ }$	$\wedge$	$\vee$	$\rightarrow, \Rightarrow$	$\leftarrow, \Leftarrow$	$\leftrightarrow, \Leftrightarrow$

Homework problems

1. Expressions, which are true for all values of variables involved, are called tautologies.
    a. Show that for any value of A, expression $A \vee \neg A$ is always true, $A \vee \neg A \Leftrightarrow T$. This tautology expresses the "law of excluded middle" meaning there is no middle ground, A must be either true or false.
    b. Similarly, show that $A \wedge \neg A \Leftrightarrow T$ is always false.
2. Another logic operation, called "exclusive or", or XOR, is defined as follows: A XOR B is true if and only if exactly one of A, B is true.
    a. Write a truth table for XOR
    b. Describe XOR using only basic logic operations AND, OR, NOT, i.e. write a formula using variables A, B and these basic operations which is equivalent to A XOR B.
3. Yet one more logic operation, NAND, is defined by $(A \text{ NAND } B) \Leftrightarrow \neg(A \wedge B)$
    a. Write a truth table for NAND
    b. What is A NAND A?
    c. * Show that you can write $\neg A$, $A \wedge B$, $A \vee B$ using only NAND (possibly using each of A, B more than once).

This last problem explains why NAND chips are popular in electronics: using them, you can build **any** logical gates.

4. Use truth tables to show that $\vee$ is commutative and associative:

$$A \wedge B \Leftrightarrow B \wedge A$$

$$A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$$

Can you show that $\wedge$ is also commutative and associative?

5. Another logic operation, called "exclusive or", or XOR, is defined as follows: A XOR B is true if and only if exactly one of A, B is true.
    a. Write a truth table for XOR
    b. Describe XOR using only basic logic operations AND, OR, NOT, i.e. write a formula using variables A, B and these basic operations which is equivalent to A XOR B.
6. Yet one more logic operation, NAND, is defined by A NAND $B \ \Leftrightarrow$ NOT (A AND B)
    a. Write a truth table for NAND
    b. What is A NAND A?
    c. Show that you can write NOT A, A AND B, A OR B using only NAND (possibly using each of A, B more than once).

This last part explains why NAND chips are popular in electronics: using them, you can build any logical gates.

7. A restaurant menu says: The fixed price dinner includes entree, dessert, and soup or salad. Can you write it as a logical statement, using the following basic pieces:

   E: *your dinner includes an entree*

   D: *your dinner includes a dessert*

   P: *your dinner includes a soup*

   S: *your dinner includes a salad*

and basic logic operations described above?

7. On the island of knights and knaves, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave. . .

You meet two people on this island, Bart and Ted. Bart claims, "I and Ted are both knights or both knaves." Ted tells you, "Bart would tell you that I am a knave." So who is a knight and who is a knave?

Logical fallacies

A fallacy is reasoning that is evaluated as logically incorrect. Fallacy vitiates the logical validity of the argument and warrants its recognition as unsound.

Formal fallacies

A formal fallacy is an error in logic that can be seen in the argument's form. All formal fallacies are specific types of *non sequiturs* (does not follow).

- Appeal to probability – is a statement that takes something for granted because it would probably be the case (or might be the case).
- Argument from fallacy – also known as fallacy fallacy, assumes that if an argument for some conclusion is fallacious, then the conclusion is false. *If you are paranoid about being stalked does not mean you are not stalked*.
- Base rate fallacy – making a probability judgment based on conditional probabilities, without accounting for the effect of prior probabilities.
- Conjunction fallacy – assumption that an outcome simultaneously satisfying multiple conditions is more probable than an outcome satisfying a single one of them.
- Masked-man fallacy (illicit substitution of identicals) – the substitution of identical designators in a true statement can lead to a false one. *I know how to solve math problems; I don't know whether this is a math problem => I don't know how to solve this problem.*
- Jumping to conclusions – the act of taking decisions without having enough information to be sure they are right.

Propositional fallacies

A propositional fallacy is an error in logic that concerns compound propositions. For a compound proposition to be true, the truth values of its constituent parts must satisfy the relevant logical connectives that occur in it (most commonly: <and>, <or>, <not>, <only if>, <if and only if>). The following fallacies involve inferences whose correctness does not follow from the properties of those logical connectives, and hence, which are not guaranteed to yield logically true conclusions.

- Affirming a disjunct – A or B; A, therefore not B.
- Affirming the consequent –if A, then B; B, therefore A.
- Denying the antecedent – if A, then B; not A, therefore not B.

Quantification fallacies

A quantification fallacy is an error in logic where the quantifiers of the premises are in contradiction to the quantifier of the conclusion.

- Existential fallacy – an argument that has a universal premise and a particular conclusion. “In a communist society everyone has everything (s)he needs”, or, “In a communist society everyone suffers from oppression”, or, “Every Unicorn has one horn on its forehead”.
- A vacuous truth is a conditional statement with a false antecedent. A statement that asserts that all members of the empty set have a certain property. For example, the statement "all students in the room are in math 9 class" will be true whenever there are no students in the room. In this case, the statement "all students in the room are not in math 9 class" would also be vacuously true, as would the conjunction of the two: "all students in the room are in Math 9 and are not in Math 9”.

Syllogistic fallacies – logical fallacies that occur in syllogisms.

- Affirmative conclusion from a negative premise (illicit negative) – when a categorical syllogism has a positive conclusion, but at least one negative premise. “Smart people don’t eat junk food. I do not eat junk food. Therefore, I a smart”.
- Fallacy of exclusive premises – a categorical syllogism that is invalid because both of its premises are negative.
- Fallacy of four terms (quaternio terminorum) – a categorical syllogism that has four terms. *Nothing is better than eternal happiness; ham sandwich is better than nothing => ham sandwich is better than eternal happiness*.
- Illicit major – a categorical syllogism that is invalid because its major term is not distributed in the major premise but distributed in the conclusion. All A are B; No C are A. Therefore, no C are B.
- Illicit minor – a categorical syllogism that is invalid because its minor term is not distributed in the minor premise but distributed in the conclusion. Pie is good. Pie is unhealthy. Thus, all good things are unhealthy.
- Negative conclusion from affirmative premises (illicit affirmative) – when a categorical syllogism has a negative conclusion but affirmative premises. All A is B. All B is C. Hence, some C is not A.
- Fallacy of the undistributed middle – the middle term in a categorical syllogism is not distributed. All Z is B; All Y is B. Therefore, all Y is Z.
- Modal fallacy – confusing possibility with necessity.

Informal fallacies

Informal fallacies – arguments that are fallacious for reasons other than structural (formal) flaws and usually require examination of the argument's content.

- Appeal to the stone (argumentum ad lapidem) – dismissing a claim as absurd without demonstrating proof for its absurdity.
- …
- Correlation proves causation (post hoc ergo propter hoc)

- Divine fallacy (argument from incredulity) – arguing that, because something is so incredible/amazing/ununderstandable, it must be the result of superior, divine, alien or paranormal agency.
- Double counting – counting events or occurrences more than once in probabilistic reasoning, which leads to the sum of the probabilities of all cases exceeding unity.
- Equivocation – the misleading use of a term with more than one meaning (by glossing over which meaning is intended at a particular time).
- …
- Psychologist's fallacy – an observer presupposes the objectivity of his own perspective when analyzing a behavioral event.
- Red herring – a speaker attempts to distract an audience by deviating from the topic at hand by introducing a separate argument the speaker believes is easier to speak to.
- Referential fallacy – assuming all words refer to existing things and that the meaning of words reside within the things they refer to, as opposed to words possibly referring to no real object or that the meaning of words often comes from how we use them.
- …