October 6, 2024 Math 8

Handout 4: Combinatorics with repetitions

Recap: main formulas of Combinatorics

- The **number of permutations**, or the number of ways to order n items is,

$$n! = n(n-1)(n-2)\dots\cdot 2\cdot 1$$

- The number of ways to choose k items out of n **if the order matters**:

$${}_nP_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\dots\cdot(n-k+1)$$

- The number of ways to choose k items out of n **if the order does not matter**:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\dots\cdot(n-k+1)}{k(k-1)(k-2)\dots\cdot 2\cdot 1}$$

The rationale behind this formula is simple. To find all possible choices of how to fill the k out of n positions with k out of n items, we first count all permutations of the given n items, which is $n!$. This counts all possible choices of k items out of n. However, the same choice is counted multiple times: for each possible choice of k items, any permutation of the remaining $n-k$ items is counted among the $n!$ permutations. Hence, we need to divide $n!$ with $(n-k)!$ to only count different choices of k items. If permutations among the k items do not matter, we also need to divide by $k!$

Combinatorics with repetitions

So far, we discussed combinations of objects (or people) where each item or person can be selected only once. The 3 people in a committee must be different persons, so from a group of 25 there are

$${}_{25}C_3 = \binom{25}{3} = \frac{23\cdot 24\cdot 25}{1\cdot 2\cdot 3}$$

combinations **without repetitions**.

Exercise. Prove that the product $2024\cdot 2023\cdot 2022\cdot\dots\cdot 2015$ is divisible by $1\cdot 2\cdot 3\cdot\dots\cdot 10$.

Now, say $n = 5$ people play a game of chance consisting of $k = 10$ rounds and in each round the winner is selected and gets a coin. If all the rounds are alike and the coins are identical, in how many ways can the coins be distributed among the players at the end? Each player can win more than one round (or none at all), and this is known as **combinations with repetitions.**

This question is equivalent to placing $k = 10$ identical marbles into $n = 5$ different jars. A common way to illustrate the solution is the “stars and bars” diagram:

1. First, place all 10 marbles ("stars") in a row; they are all the same, so the order does not matter.
2. Then, divide this row into $n = 5$ groups by inserting $(5 - 1) = 4$ dividers ("bars") anywhere in the row; the "bars" are also identical.
3. Counting from the left, each group of stars is the number of marbles for the jar; if two "bars" are next to each other, then the corresponding jar will be empty.

Below is an illustration for $k = 10$ and $n = 5$:

$$\overbrace{**********}^{n\ stars} = \overbrace{* \mid *** \mid ** \mid\mid ****}^{(n+k-1) stars and\ bars}$$

All possible permutations of n identical "stars" and $(k - 1)$ "bars" give you all possible ways to split into k groups (jars). We counted such permutations in the previous class:

$$(Permutations\ of\ n\ stars, k - 1\ bars) = \frac{(k + n - 1)!}{n!\,(k - 1)!} = \binom{k + n - 1}{k - 1} = \binom{k + n - 1}{n}$$

So, in our example, the number of ways to put 10 marbles into 5 jars (or split 10 coins between 5 players) is,

$$(10\ marbles\ into\ 5\ jars) = \binom{10 + 5 - 1}{10} = \frac{14!}{10!\,4!}$$

Exercises

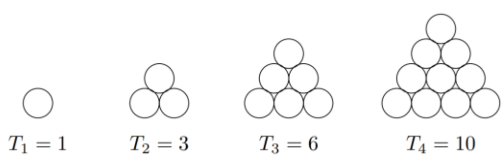
1. How many ways are there to place 20 passengers in 3 buses? (each bus can seat at least 20).
2. How many ways are there to split number 10 into a sum of 3 positive integers (order matters)? For example,

$$10 = 8 + 1 + 1;\ 7 + 2 + 1;\ 7 + 1 + 2; \ldots.$$

3. In a game of 3D chess played in a 8x8x8 cube board, there is a new figure called "squire". It can move only into one of 8 adjacent non-diagonal cubes. If a squire starts from the corner cube and wants to get away from it as far as possible, in how many positions can it end up after 6 moves?

Homework assignment

1. Calculate how many ways are there
   a. to distribute 5 identical marbles into 3 different jars?
   b. to distribute 3 identical marbles into 5 different jars; can you compute it differently?
   c. stack 10 identical coins in a row of 4 stacks, where each stack must have at least 1 coin?
2. You have a large well-mixed bag of marbles of four different colors, otherwise identical. How many combinations (numbers of marbles of each color) are possible if you randomly pull 4 marbles from a bag? 5 marbles? 6 marbles?
3. How many five-letter "words" are there with the letters in alphabetical order? (Here a "word" is any sequence of letters from the alphabet, "a" through "z".)
4. Imagine that you have a regular cubic dice with numbers 1 to 6 on the sides. You roll the dice a few times and write the combination down. If the order does not matter, how many different combinations can you get when rolling 3 times? 5 times? 10 times?
5. A monomial is a product of powers of variables, i.e. an expression like x^3y^7.
   a. How many monomials in variables x, y of total degree of exactly 10 are there? (Note: this includes monomials which only use one of the letters, e.g. x^{10}.)
   b. Same question about monomials in variables x, y, z.
   c. What about 4-th degree monomials in variables x, y, z, t?
   d. How many monomials in variables x, y of degree at most 15 are there?
6. How many different monomials in 3 variables x, y, z of total degree n are there? in 4 variables?
7. Each student in a class of 20 must pick one day of the week (Monday through Friday) to make a 10-minute presentation. Depending on their choices, the teacher must make the schedule reserving a fraction of each day for the presentations. How many different schedules are possible (without regard to who and when makes the presentations)?
8. Let T_n be the number of circles in a triangular shape with n levels like the ones below (these are sometimes called triangular numbers):



   a. Note that $T_3 = T_2 + 3$, and $T_4 = T_3 + 4$. Is it true that in general, $T_n = T_{n-1} + n$? Why or why not?
   b. Look at Pascal triangle. Can you find these numbers there?
   c. Can you write a general formula for T_n?
9. * What if instead of drawing circles on plane, we were arranging balls in a pyramid? Can you guess how many balls we would have in pyramid with 1 level? with 2, 3, 4 levels? Can you find these numbers in Pascal triangle?
10. Using binomial formula, prove that for any integer $n \geq 0$, $\forall\, x \geq 0$, $(1 + x)^n \geq 1 + nx$
11. * Which number is greater, 2024^{2025}, or 2025^{2024}?