September 29, 2024 Math 8

Handout 3: ${}_nC_k$ and Newton's binomial

Recap: main formulas of Combinatorics

- The **number of permutations**, or the number of ways to order n items is,

$$n! = n(n-1)(n-2)\ldots\cdot 2\cdot 1$$

- The number of ways to choose k items out of n **if the order matters**:

$${}_nP_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\ldots\cdot(n-k+1)$$

- The number of ways to choose k items out of n **if the order does not matter**:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\ldots\cdot(n-k+1)}{k(k-1)(k-2)\ldots\cdot 2\cdot 1}$$

The rationale behind this formula is simple. To find all possible choices of how to fill the k out of n positions with k out of n items, we first count all permutations of the given n items, which is $n!$. This counts all possible choices of k items out of n. However, the same choice is counted multiple times: for each possible choice of k items, any permutation of the remaining $n-k$ items is counted among the $n!$ permutations. Hence, we need to divide $n!$ with $(n-k)!$ to only count different choices of k items. If permutations among the k items do not matter, we also need to divide by $k!$

Properties of ${}_nC_k$ and Pascal's triangle

${}_nC_k$ satisfies the following recursive relation:

$${}_nC_k = {}_{n-1}C_k + {}_{n-1}C_{k-1}$$

Using this recurrency rule, one can fill up the Pascal's triangle where the k-th entry in n-th line is exactly ${}_nC_k$ (both n and k are counted from 0, not from 1). Every number is obtained by adding the two nearest numbers above,

Row #	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

Problem. What is the sum of all numbers in the n-th row of Pascal's triangle? Can you guess the pattern – and once you guessed it, can you justify your guess?

This problem illustrates the method that we used for solving various random walk problems. The method consists in mapping an n-step walk onto an n-character "word" made of two letters. In the drunkard's problem, a walk is mapped onto a sequence of letters like "NNSNSSSNNSNSNNN" where the letters "N" and "S" denote a step made to the North or to the South, respectively. Another way to look at this, is to represent such "word" (or a walk) by a binary sequence of zeros and ones. For an n-step walk, the total number of possibilities is 2^n. Now, consider the following questions:

1. How many walks are there with exactly k steps to the North?
2. How many possible outcomes of the random walk are there? What are these outcomes?
3. What do you obtain by adding all these possible outcomes?

Exercise. Each strand of a DNA molecule is a sequence of four bases, usually denoted by letters A, C, G, and T. Hence, the entire DNA sequence is described by a word like "AGGCCTAATTCAGGG". Consider the following questions.

1. How many DNA molecules of length n can there be?
2. How many DNA molecules of length n can there be which have equal amounts of each base?
3. How many DNA molecules of length n can there be in which the amounts of the bases A, C, G, and T are n_A , n_C , n_G , and n_T, respectively?

Newton's binomial formula

The binomial coefficients, ${}_nC_k$, are so called because they have important application in Newton's binomial expansion, or **binomial formula**,

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{k} a^{n-k}b^k + \cdots + \binom{n}{n} b^n$$

A proof of the above formula can be obtained by expanding the n-th power as a product of the n parentheses, and then counting all terms of the type $a^{n-k}b^k$, $0 \le k \le n$ in the expansion,

$$\begin{aligned}(a+b)^n &= \underbrace{(a+b)(a+b)\dots(a+b)}_{n\ times} \\ &= N_0a^n + N_1a^{n-1}b + N_2a^{n-2}b^2 + \cdots + N_ka^{n-k}b^k + \cdots + N_nb^n\end{aligned}$$

to obtain $N_k = \binom{n}{k} = {}_nC_k$.

Homework assignment

In all the problems, you can write your answer as a combination of factorials, nCk, and other arithmetic – you do not have to do the computations. As usual, please write your reasoning, not just the answers!

1. Use the binomial formula to expand the following expressions:
   a. $(x - y)^3 =$
   b. $(a + 3b)^3$
   c. $(2x + y)^5 =$
   d. $(x + 2y)^6 =$
2. Find the coefficient of x^8 in the expansion of $(2x + 3)^{14}$
3. Use binomial formula to:
   a. Compute $\left(1+\sqrt{3}\right)^6 + \left(1-\sqrt{3}\right)^6$e
   b. Show that $\left(1+\sqrt{3}\right)^{12} + \left(1-\sqrt{3}\right)^{12}$ is an integer
4. Compute $(x + 2y)^6 - (x - 2y)^6$
5. Use the binomial formula to compute
   a. Sum of all numbers in the n-th row of Pascal's triangle. [Hint: use the binomial formula for $a = b = 1$]
   b. Alternating sum of all numbers in the n-th row of Pascal's triangle:

   $$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} + \cdots + (-1)^n \binom{n}{n}$$

   Can you find a way of answering thsi question without using the binomial formula?
6. Let p be prime.
   a. Show that each of the binomial coefficients$\binom{p}{k}$, $1 \le k \le p-1$ is divisible by p
   b. Show that if a, b are integers, then $(a + b)^p - a^p - b^p$ is divisible by p.
7. Long ago, the four nations decided to hold a relay race competition. Forty-eight people signed up, twelve from each of four element-nations: Water, Earth, Fire, Air. However, a relay run consists of four people, so only sixteen of those people can compete.
   a. Given that each nation must select four people to form a team, in how many ways can this be done?
   b. Now consider they run the competition slightly differently: teams will consist of one person from each nation (4 total), and four teams will be chosen. How many ways can this be done?
8. [You can use a calculator (or Wolfram Alpha) for this problem.]
   a. Given a group of 25 people, we ask each of them to choose a day of the year (non-leap, so there are 365 possible days). How many possible combinations can we get? [Order matters: it is important who has chosen which date]

b. The same question, but now we additionally require that all chosen dates be different.

c. In a group of 25 people, what are the chances that no two of them have their birthday on the same day? Conversely, what is the chance that at least two people have the same birthday?