September 22, 2024 Math 8

Handout 2: Combinatorics and Pascal's triangle

Recap: main formulas of Combinatorics

- The **number of permutations**, or the number of ways to order n items is,

$$n! = n(n-1)(n-2)\dots\cdot 2\cdot 1$$

- The number of ways to choose k items out of n **if the order matters**:

$$_nP_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\dots\cdot(n-k+1)$$

- The number of ways to choose k items out of n **if the order does not matter**:

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\dots\cdot(n-k+1)}{k(k-1)(k-2)\dots\cdot 2\cdot 1}$$

Properties of $_nC_k$ and Pascal's triangle

Exercise. Verify that $_nC_k$ satisfies the following recursive relation:

$$_nC_k = {}_{n-1}C_k + {}_{n-1}C_{k-1} \quad or, equivalently, \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Let us fill up a table of triangular shape, starting with the number 1 in the top row #0 and using the rule that every number is obtained by adding the two nearest numbers above,

Row #	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

We can note that the k-th entry in n-th line is exactly $_nC_k$. Note that both n and k are counted from 0, not from 1 (see Table). For example, $_4C_2 = 6$ is in the middle of the 4th row counting from 0, and $_0C_0 = 1$. This is clear from the recurrency defining these numbers,

$$_0C_0 = {}_nC_0 = {}_nC_n = 1$$

$$_nC_k = {}_{n-1}C_k + {}_{n-1}C_{k-1}, \qquad 1 \le k \le n-1$$

The above recurrency formula is illustrated by the following problem.

Problem. Suppose I have a stack of n cards including a joker and I am drawing k cards from that stack. If the joker is among the k cards that I draw, I loose; otherwise, I win. Let $M(n;\ k)$ be the number of ways I can draw k cards. Let a denote the number of ways I can win (that is, the number of draws not containing joker), and b denote the number of draws including the joker, where I loose.

1. Prove that $a + b = M(n;\ k)$
2. Prove that $a = M(n-1;\ k)$, and $b = M(n-1;\ k-1)$
3. Deduce from the above that $M(n;\ k) = M(n-1;\ k) + M(n-1;\ k-1)$

Homework assignment

In this homework assignment (as in other assignments in this class), many problems are challenging and require some thought. Try to start early. You are not expected to be able to solve all of the problems, so do not be discouraged if you can't solve some of them. The solutions are to be submitted through Google classroom. Please make sure that you show not just the answer but also the solution, i.e. your reasoning explaining how you arrived at the answer. Ideally, your solution should be such that someone who doesn't know how to solve this problem can read it and follow your arguments. Please submit your work and thoughts even if your solution is incomplete – your work might still deserve partial credit.

It is enough if you can write the answers in terms of factorials and binomial coefficients - it is not necessary to compute the actual numbers: an answer like 13! or ${}_{10}C_5$ is good enough.

1. If we want to choose a president, vice-president, and two assistants from a 15-member club, in how many ways can we do this?
2. 5 kids come to a store to choose Halloween costumes. The store sells 25 different costumes. Assuming the store has enough stock for the kids to choose the same costume if they want, in how many ways can the kids choose the costumes? What if they want to choose so that all costumes are different?
3. Suppose I flip a coin three times, each time recording the outcome (for example, the coin may land heads then tails then heads, which I will write as HTH, where order matters). I will refer to this three-letter combination as the result - for example, HHH is the only result that has no tails.
    a. How many results are there with exactly one tail?
    b. How many results are there with exactly two tails?
4. Five octopuses are working at the beach's local landfood restaurant. They want to assign lunch shifts, so that some of the octopuses can have lunch from 12:00pm to 1:00pm, and the others can have lunch from 1:00pm to 2:00pm.
    a. If they decide to have two octopuses take the first shift and three take the second, how many possible ways are there to assign shifts?
    b. If the noontime hour is especially busy and they decide to have just one octopus take the first shift and the remaining four take the second shift, how many ways are there to assign shifts?
5. Are there any rows in the Pascal triangle where all numbers are odd? Which rows are these?
6. What is the sum of all numbers in the n-th row of Pascal's triangle? Can you guess the pattern — and once you guessed it, justify your guess?
7. How many ways are there to place two rooks on a chessboard so that they are not attacking each other? [For those of you who are unfamiliar, a chessboard is an 8 x 8 grid of squares, and rooks are pieces that can occupy any one of these individual squares and may attack any other piece that's in the same row or column of the board as itself.]

8. How many different paths are there on a 6 x 6 chessboard, connecting the lower left corner with the upper right corner such that
    a. The path always goes to the right or up, never to the left or down
    b. The path never goes below the diagonal (being on diagonal is OK)

    Hint: each cell can be reached from the ones to the left (except the first column) or below (except the diagonal). Count the total number of paths leading to each accessible cell.