Math test
March 9, 2025

Main Algebraic Identities/formula

$$a^{-n} = \frac{1}{a^n}$$

$$(a^m)^n = a^{mn}$$

$$\frac{m}{n} = \sqrt[n]{a^m}$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$a^2 - b^2 = (a-b)(a+b)$$

Arithmetic series

$$a_n = a_1 + (n-1)d$$

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

$$d = \frac{a_s - a_t}{s - t}$$

$$S = \frac{(a_1 + a_n) \times n}{2}$$

Geometric series

$$a_n = a_1 \times q^{n-1}$$

$$a_n = \sqrt{a_{n-1}.\, a_{n+1}}$$

$$S_n = a_1 \times \frac{(1-q^n)}{1-q}$$

$$\mathrm{S} = \frac{a_1}{1-q}$$

Binomial coefficients

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (***order doesn't matter***)

- Formula for binomial coefficients

There is an explicit formula to calculate $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

- Formula for permutations (the number of ways of choosing k items out of n when ***the order matters***):
  Compare it with the number of ways of choosing k items out of n when the order matters:

$$_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

Binomial probabilities

The binomial coefficients are also useful in calculating probabilities. Imagine that we have some event that happens with probability p (“success”) and does not happen with probability $q = 1 - p$ (“failure”). Then the probability of getting k successes in n trials is:

$$P(k\ successes\ in\ n\ trials) = \binom{n}{k} p^k q^{n-k}$$

Where,
• p — probability of success in one try;
• q = 1 – p — probability of failure in one try;
• n — number of trials;
• k — number of successes;
• n – k — number of failures.

Coordinate geometry

The **midpoint** M of a segment AB with endpoints A(x1, y1) and B(x2, y2) has coordinates:

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

The distance between two points $A(x1, y1)$ and $B(x2, y2)$ is given by the following formula:

$$\boldsymbol{d} = \sqrt{(\boldsymbol{x_2} - \boldsymbol{x_2})^2 + (\boldsymbol{y_2} - \boldsymbol{y_1})^2}$$

Line: $y = \boldsymbol{m}x + \boldsymbol{b}$
with a **slope** $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ and intercept ***b.***
Parabola: $y = ax^2 + bx + c$ (standard form) or $y = a(x-h)^2 + \mathrm{k}$ (vertex form)
Circles: The equation of the circle with the center $M(x0, y0)$ and radius ***r*** is:$(x - x0)^2 + (y - y0)^2 = r^2$.

Problems:

1. Find the coordinates of the points where the circle $(x+2)^2+(y-4)^2=5$ meets the line $y=-2x+4$.
2. Write equation of a line passing through point (4,4) and parallel to line y=7/4 x -4. What is the equation is the line in perpendicular? (Hint: parallel lines have same slope whereas for perpendicular lines, the product of slope is -1).
3. Draw graph of function y=|x-1|. Then perform mirror operations on x axis and y axis (i.e change sign of x and y) and plot the corresponding functions.
4. Expand as sums of powers of x : $(2x+5)^5$
5. Plot the following functions and determine the region where the function is less than zero [Hint: determine the vertex i.e. write in the form of $y= (x-h)^2 + k$ by completing the equation. Also find the roots i.e. value of x when y=0. This will help you plot the function].
   a. $y = x^2 + 2x + 3$
   b. $y = -x^2 + 6x - 9$

6. Factorize: (i.e., write as a product) the following expressions:
   a. $p^4 + 4z^{4n}$
   b. $t^2 - 3/2\ t + 1/2$

7. (a) Solve inequality: $|x-2| > 3$
   (b) An arithmetic progression has first term $a_1 = a$ and common difference $d = -1$. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.

8. Calculate the sum: $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$
   What is the sum if the terms of the series continue up to infinity?
9. Solve the following inequalities:
   (a) $(x+3)(x-2)^2 \leq 0$
   (b) $\frac{x-2}{(x+3)} \leq 3$ [Hint: convert division into multiplication problem.]

10. If we toss a coin 10 times, what is the probability that all will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?
11. Find p such that the sum of squares of the roots ($x_1^2 + x_2^2$) of the equation $x^2 - px + p + 7 = 0$ equal to 10 [Hint: use Vieta's formula.]