Homework 12: Inequalities – interval method, quadratic functions - vertex form

1. Converting from standard to vertex form [The following technique is useful to plot quadratic function which will help you to solve inequalities. I did not get time to teach you this. If you find it difficult, you may use graphing tool like Desmos or graphing calculator to visualize graph.]

We know how to draw the graph of $y = x^2$. It's a parabola.

- We know that the graph of $y = x^2 + 2$ can be obtained from the graph of $y = x^2$ by shifting up by +2 units (or down, if $y = x^2 - 2$)
- We know that the graph of $y = (x + 5)^2$ can be obtained from the graph of $y = x^2$ by shifting left by 5 units (or right, if $y = (x - 5)^2$).
- Based on the two facts above, we can draw a graph of any function of the type $y = (x - h)^2 + k$. The values (h, k) are the coordinates of the vertex of the parabola.

In general. we can transform any quadratic function $\boldsymbol{y = ax^2 + bx + c}$ to $\boldsymbol{y = a(x - h)^2 + k}$.
This is called transforming from standard into vertex form. The coefficient a has the same value in both forms. Note that, if $b = 0$, then your equation in standard form is already in a vertex form with vertex coordinates $(0, k = c)$.

We will use two ways to convert a quadratic function from standard into vertex form:

- **Method 1: completing the square**. We have learned how to do this using the formulas for fast multiplication.
  Example: $y = 2x^2 + 4x - 2 = 2[\boldsymbol{x^2 + 2x} - 1] = 2\left[x^2 + 2.x.1 + 1^2 - 1^2 - 1\right] = 2[(x + 1)^2 - 1 - 1] = 2[(x + 1)^2 - 2] = 2(x + 1)^2 - 4.$
- **Method 2: find the vertex**. Determine the coefficients a, b, c. Find the vertex *x-coordinate* $\boldsymbol{x_v = h = -\frac{b}{2a}}$.
  Then, substitute $\boldsymbol{x_v}$ in the equation you are converting and solve for y, $y = a\boldsymbol{x_v}^2 + b\boldsymbol{x_v} + c$. The found value is the vertex *y-coordinate*, $\boldsymbol{y_v = k}$. Write the equation in a vertex form $y = a(x - h)^2 + k$.
  Example: $y = 2x^2 + 4x - 2, \quad a = 2, \; b = 4, \; c = -2$

  Vertex *x-coordinate:* $x_v = h = -\frac{b}{2a} = -\frac{4}{2.2} = -1$

  Vertex *y-coordinate:* $y_v = 2x_v^2 + 4x_v - 2 = 2(-1)^2 + 4(-1) - 2 = 2 - 4 - 2 = -4, \quad k = -4$

  New function: to $y = a(x - \boldsymbol{h})^2 + \boldsymbol{k} = 2(x + 1)^2 - 4$

2. Solving polynomial inequalities using the interval method

So far, we have solved quadratic and rational inequalities using linear inequalities. We can also consider polynomial inequalities: they would have terms like $x^2; x^3$,etc. The general rule for solving polynomial inequalities is as follows:

- Find the roots and factor your polynomial, writing it in the form $p(x) = a(x - x_1)(x - x_2)$ (for polynomial of degree more than 2, you would have more factors).
- With the roots $x_1; x_2; \ldots$: divide the real line into intervals; starting with the first interval, choose a number from that interval to be your x, substitute it in the factored inequality, and determine the sign of each factor in your inequality. Then determine the sign of the product of all factors. Repeat for each interval.
- If the inequality has ≥ or ≤ signs, you should also include the roots themselves into the intervals.
- The intervals whose signs match the sign of the inequality are your solutions.

Example 1. $x^2 + x - 2 > 0$

Solution. We find the roots of the equation $x^2 + x + 2 = 0$ to be $x = -2;\ 1$. The inequality in factored form becomes $(x + 2)(x - 1) > 0$, and the roots $-2, 1$ divide the real line into three intervals $(-\infty; -2); (-2; 1); (1; +\infty)$.
It is easy to see that the polynomial $x^2 + x + 2$ is positive on the first and the third intervals and negative on the second one. The solution of the inequality is then all x in interval one and three ($x < -2 \; OR \; x > 1$). We write this also as $x \in (-\infty; -2) \cup (1; +\infty)$. (sign ∪ means "or"). Solving polynomial inequalities of second and higher order makes us realize that if we determine the sign of the first interval, the signs of the following intervals alternate. The graph of the polynomial crosses the $x - axis$ from above (" +" interval), goes below (" − " interval), … the curve "snakes" around the axis when crossing the roots. This is why this method for solving polynomial inequalities is also known as "snake" method. ***Careful*** – if a factor is raised at an even power, the sign will always be positive and the alternation will not apply.

Homework problems

1. Convert the quadratic functions into vertex form using completing the square method (Method 1). **Sketch** each graph's vertex position and its parabola shape next to the function. You may take help from graphing calculator or an online graphing tool.
   a. $y = x^2 + 2x - 3$
   b. $y = x^2 + 2x + 3$
   c. $y = -x^2 + 6x - 9$
   d. $y = 3x^2 + x - 1$

2. Convert the quadratic functions into vertex form using finding the vertex method. [I did not teach you this in class but the above notes should help you.]
   a. $y = x^2 + 2x - 3$
   b. $y = x^2 + 2x + 3$
   c. $y = -x^2 + 6x - 9$
   d. $y = 3x^2 + x - 1$

3. Solve the quadratic equations. Use the roots to convert the inequalities into factored form. Draw the roots on the x-axis and solve the corresponding inequalities using the **interval method**. Write the solutions for the inequalities using intervals. **Example:** for $x^2 + 2x - 3 > 0 \quad x \in \ldots$ .Note: these are the same functions as in problems 1 and 2. Sketch again each graph (the parabola): color with red the parts of the graph where $y > 0$, color with blue the parts of the graph where $y < 0$.
   a. $x^2 + 2x - 3 = 0$; $x^2 + 2x - 3 > 0$;
   b. $x^2 + 2x + 3 = 0$; $x^2 + 2x + 3 > 0$;
   c. $-x^2 + 6x - 9 = 0$; $-x^2 + 6x - 9 > 0$;
   d. $3x^2 + x - 1 = 0$; $3x^2 + x - 1 > 0$

4. The signs of the inequalities are inverted; the problem is asking for what values of x the y-values of the functions are negative. You have already converted to factored forms. Draw the roots on the x-axis and solve the corresponding inequalities using the **interval method**. Write the solutions for the inequalities using intervals.
   Look at the graphs you sketched in problem 3. Do the blue parts correspond to your solutions?
   a. $x^2 + 2x - 3 < 0$
   b. $x^2 + 2x + 3 < 0$
   c. $-x^2 + 6x - 9 < 0$;
   d. $3x^2 + x - 1 < 0$