Homework 9: Intro to quadratic equation

1. Quadratic equation in a standard form.

Today we discussed how one solves quadratic equation, starting from the **standard form**: $ax^2 + bx + c = 0$
A quadratic equation could have no solution, one solution, or two solutions depending on the coefficients a, b, and c.

We could solve such an equation by presenting it in a **factored form**: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots.* The factored form will also help us find a general formula for solving any quadratic equation using the coefficients a, b, c.

2. Solving the incomplete quadratic equation by factorizing.

- When $c = 0$, $ax^2 + bx = 0$

To solve, factorize as $x(ax + b) = 0$ and the two terms in the product to be equal to zero. The two roots are $x_1 = 0$ and $x_2 = -b/a$

- When $b = 0$, $ax^2 + c = 0$

If c < 0, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a - b)(a + b)$. (*)
For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$. Setting each term in the product to zero gives solutions of $+5$ and -5.

If c > 0, there are no real solutions. An easy way to see this is to solve directly for x: $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$; No number squared is equal to a negative number!

2. Solving the complete quadratic equation

- <u>By completing the square</u>

"Completing the square" works by using the formulas for fast multiplication $(a \pm b)^2 = a^2 \pm 2ab + b^2$ $(*)$
Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x+3)^2 - 7 = (x+3)^2 - \left(\sqrt{7}\right)^2 = \left(x + 3 + \sqrt{7}\right)\left(x + 3 - \sqrt{7}\right)$$

Thus, $x^2+6x+2 = 0$ if and only if $(x+3+\sqrt{7}) = 0$, which gives $x = -3-\sqrt{7}$, or $(x+3-\sqrt{7}) = 0$, which gives $x = -3+\sqrt{7}$.

- <u>By using the quadratic formula</u>

Completing the square works in general for any quadratic equation in a standard form
If *a* = 1, then:

$$x^2 + bx + c = x^2 + 2\frac{b}{2}x + c = \left(x^2 + 2\frac{b}{2}x + \frac{b^2}{2^2}\right) - \frac{b^2}{2^2} + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{2^2} = \left(x + \frac{b}{2}\right)^2 - \frac{D}{2^2} \qquad \text{eq (1)}$$

Thus $x^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2}\right)^2 = \frac{D}{4}$

If $a \neq 1$, then: $ax^2 + bx + c = 0$ is equivalent to: $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$, where $\boldsymbol{D = b^2 - 4ac}$

The determinant D determines the number of solutions. D < 0, there are no real solutions; if D = 0, there is one solution,

if *D* > 0, the solutions are:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{D}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{D}}{2a} \qquad \text{eq (2)}$$

<u>Homework problems</u>

(*) The parameters a and b in the formulas for fast multiplication $a^2 - b^2 = (a - b)(a + b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$ <u>are not the same</u> as the coefficients a, b, and c used in the standard form of the quadratic equation!

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.
Note: Use the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

1. This problem requires that you carefully check your work and think:
   a. Use formula (1) to prove that for any x, $x^2 + bx + c \geq -D/4$, with equality only when $x = -b/2$.
   b. Find the minimal possible value of the expression $x^2 + 4x + 2$ [Hint: use part a) or complete the square]
   c. Given a number *a* > 0, find the maximal possible value of the expression $x(a - x)$ (the answer will depend on the value or values of a. In this case, a is called a *parameter*).

2. Derive Eq. 2 (given in previous page) for the most general quadratic equation $ax^2 + bx + c = 0$.

3. Convert the following equations to standard form (open brackets). Determine the coefficients a, b, and c. Do not solve the equations!
   a. $2(x-3)(x-1) = 0$
   b. $(x-2)^2 + (2x+3)^2 = 13 - 4x$
   c. $(x-4)(x+4) = 1$

4. Solve the following quadratic equations by converting to factorized form.
   a. $2x^2 - 3x = 0$
   b. $x^2 - 15 = 1$
   c. $3x^2 - 9 = 0$
   d. $2(x-3)(x-1) = 0$

5. Complete the square and find the solutions for the following quadratic equations:
   a. $x^2 + 4x + 3 = 0$
   b. $y^2 + 4y - 5 = 0$

6. Solve the following equations. Carefully think what method you will use and write all steps in your argument. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (2)?
   a. $x^2 - 5x + 5 = 0$
   b. $\frac{x}{x-2} = x - 1$
   c. $x^2 = 1 + x$
   d. $2x(3 - x) = 1$
   e. $x^3 + 4x^2 - 45x = 0$

7. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2} = 7$ and $x^3 + \frac{1}{x^3}$ [Hint: try completing the square, completing the cube …]

8. (*) Consider the sequence $x_1 = 1$, $x_2 = \frac{x_1}{2} + \frac{1}{x_1}$, $x_3 = \frac{x_2}{2} + \frac{1}{x_2}$ . . . .

Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you ***guess*** this value? [Hint: solve equation $x = \frac{x}{2} + \frac{1}{x}$]

(*) The parameters a and b in the formulas for fast multiplication $a^2 - b^2 = (a-b)(a+b)$, $(a \pm b)^2 = a^2 \pm 2ab + b^2$ are not the same as the coefficients a, b, and c used in the standard form of the quadratic equation!