MATH 7: HOMEWORK 11
Introduction to quadratic equation.
January 5, 2025

1. Quadratic equation in a standard form.

Today we discussed how one solves quadratic equations, starting from the **standard form**: $ax^2 + bx + c = 0$
A quadratic equation could have no solution, one solution, or two solutions depending on the coefficients a, b, and c.

We could solve such an equation by presenting it in a **factored form**: $(x - x_1)(x - x_2) = 0$, where x_1 and x_2 are the solutions of the equation, also known as *roots.* The factored form will also help us find a general formula for solving any quadratic equation using the coefficients a, b, c.

2. Solving the incomplete quadratic equation by factorizing.

- When $c = 0$, $ax^2 + bx = 0$

To solve, factorize as $x(ax + b) = 0$ and the two terms in the product to be equal to zero. The two roots are $x_1 = 0$ and $x_2 = -b/a$

- When $b = 0$, $ax^2 + c = 0$

If c < 0, factorize the equation using the formula for fast multiplication $a^2 - b^2 = (a - b)(a + b)$. (*)
For example, $x^2 - 25 = 0 \Rightarrow x^2 - 5^2 = 0 \Rightarrow (x - 5)(x + 5) = 0$. Setting each term in the product to zero gives solutions of $+5$ and -5.

If c > 0, there are no real solutions. An easy way to see this is to solve directly for x: $x^2 + 25 = 0 \Rightarrow x^2 = -5^2$; No number squared is equal to a negative number!

3. Solving the complete quadratic equation

- <u>By completing the square</u>

"Completing the square" works by using the formulas for fast multiplication $(a \pm b)^2 = a^2 \pm 2ab + b^2$ (*)
Here is an example how to rewrite the standard form of an equation to factorized form by completing the square:

$$x^2 + 6x + 2 = x^2 + 2 \cdot 3x + 9 - 9 + 2 = (x + 3)^2 - 7 = (x + 3)^2 - \left(\sqrt{7}\right)^2 = \left(x + 3 + \sqrt{7}\right)\left(x + 3 - \sqrt{7}\right)$$

Thus, $x^2+6x+2 = 0$ if and only if $(x+3+\sqrt{7}) = 0$, which gives $x = -3-\sqrt{7}$, or $(x+3-\sqrt{7}) = 0$, which gives $x = -3+\sqrt{7}$.

- <u>By using the quadratic formula</u>

The determinant $\boldsymbol{D = b^2 - 4ac}$ determines the number of solutions. If D < 0, there are no real solutions; if D = 0, there is one solution, if *D* > 0, the solutions are: $x + \frac{b}{2a} = \pm\sqrt{\frac{D}{4a^2}}$

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

$$D = b^2 - 4ac$$

eq (1)

Homework problems

1. Convert the following equations to standard form (open brackets). Determine the coefficients a, b, and c. Do not solve the equations!
   a. $2(x-3)(x-1) = 0$
   b. $(x-2)^2 + (2x+3)^2 = 13 - 4x$
   c. $(x-4)(x+4) = 1$

2. Solve the following quadratic equations by converting to factorized form.
   a. $2x^2 - 3x = 0$
   b. $x^2 - 15 = 1$
   c. $3x^2 - 9 = 0$
   d. $2(x-3)(x-1) = 0$

3. Complete the square and find the solutions for the following quadratic equations:
   a. $x^2 + 4x + 3 = 0$
   b. $y^2 + 4y - 5 = 0$

4. Solve the following equations. Carefully think what method you will use and <u>write all steps</u> in your argument. The following questions may help you: is the equation in a standard or in a factored form?; what are the coefficients a, b, c? Are some of these coefficients zero? Shall I factorize or use the quadratic formula from eq (1)?
   a. $x^2 - 5x + 5 = 0$
   b. $\frac{x}{x-2} = x - 1$
   c. $x^2 = 1 + x$
   d. $2x(3 - x) = 1$
   e. $x^3 + 4x^2 - 45x = 0$

5. If $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$ and $x^3 + \frac{1}{x^3}$ [Hint: try completing the square, completing the cube …]

6. In the 12th century, Indian mathematician Bhaskara formulated the following problem. Solve it! (translated from original text)
   Out of a party of monkeys, the square of one fifth of their number diminished by three went into a cave. The one remaining monkey was climbing up a tree. What is the total number of monkeys?

7. (*) Consider the sequence $x_1 = 1,\ x_2 = \frac{x_1}{2} + \frac{1}{x_1},\quad x_3 = \frac{x_2}{2} + \frac{1}{x_2}\ \ \ldots.$

Compute the first several terms; does it seem that the sequence is increasing? decreasing? approaching some value? If so, can you ***guess*** this value? [Hint: solve equation $x = \frac{x}{2} + \frac{1}{x}$]