MATH 7: HOMEWORK 9
BINOMIAL PROBABILITIES.
November 17, 2024

1. Summary: binomial coefficients

We discussed binomial coefficients and saw that they provide answers to the following questions:

$_nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right
= the number of words that can be written using k ones and n − k zeroes
= the number of ways to choose k items out of n (***order doesn't matter***)

- Formula for binomial coefficients

There is an explicit formula to calculate $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!}$$

- Formula for permutations (the number of ways of choosing k items out of n when ***the order matters***): Compare it with the number of ways of choosing k items out of n when the order matters:

$$_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

Compare the two ways of choosing: the number of ways of choosing k items out of n when the order doesn't matter and when the order matters:

For example, there are $5 \cdot 4 = 20$ ways to choose two items out of 5 if the order matters, and $\frac{5 \cdot 4}{2} = 10$ if the order does not matter.

2. Binomial probabilities

The binomial coefficients are also useful in calculating probabilities. Imagine that we have some event that happens with probability p ("success") and does not happen with probability $q = 1 - p$ ("failure"). Then the probability of getting k successes in n trials is:

$$P(k\ successes\ in\ n\ trials) = \binom{n}{k} p^k q^{n-k}$$

Where,
• p — probability of success in one try;
• q = 1 – p — probability of failure in one try;
• n — number of trials;
• k — number of successes;
• n – k — number of failures.

Example: You roll a die 100 times. What is the probability of getting a 6 exactly 20 times?
Solution: Here we roll the die n = 100 times, we got a 6 k = 20 times, where the probability for rolling a 6 is p = 1/6, and the probability for not rolling a 6 is q = 5/6. Then using the binomial probability formula, we calculate the probability as:

$$P = \binom{100}{20} \left(\frac{1}{6}\right)^{20} \left(\frac{5}{6}\right)^{80}$$

Homework problems

Note: In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use the Pascal's triangle to find the value of $\binom{n}{k}$, where k is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. (a) There are 15 students in a soccer club. The coach needs to select 11 of them to form the team for a match against another club. How many possibilities does he have?

(b) There are 15 students in a soccer club. The coach needs to select a goalkeeper and 10 players to form the team for a match against another club. How many possibilities does he have?

(The difference between two parts is that in the first case, the coach needs to select 11 players —no need to specify their positions. In the second part, he needs to select 11 players and specify which of them will be the goalkeeper.)

2. In one of the lotteries run by New York State, "Sweet Million", they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win $1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]

(a) How many ways are there to choose 6 numbers out of 40?

(b) What are your chances of winning?

(c) If a lottery ticket cost $1, how much money does New York State make for each ticket sold (on average)?

Bonus question: find online the rules for another NY lottery, "Mega Millions", and analyze your chances to win.

3. In poker, players are drawing "hands" (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).

(a) How many possible hands are there?

(b) What are your chances of drawing a hand in which all cards are spades?

(c) What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]

(d) What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]

4. A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$.

(a) The hunter makes 5 shots. What is the probability that she misses all five?

(b) What is the probability that out of 5 shots, she will hit a duck at least once? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)

(c) What is the probability that out of 5 shots, she will hit exactly once? Will this double if she makes 10 shots?

(d) What is the probability that out of 5 shots, she will hit a duck exactly three times? Will this probability double if she makes 10 shots? (You can use the calculator for computing the answers)

(e) What is the probability that she hits a duck half times or more if she fires 5 times (that is, 3, 4, or 5 hits)? What about if she fires 10 times (that is 5, 6, 7, 8, 9, or 10 hits)?

(f) What is the most likely number of hits out of 5 shots? And out of 10 shots?

5. At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win a stuffed toy (worth about $5). Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.

(a) If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?

(b) Same questions for five balls.

(c) What about seven balls?

(d) How much should the organizers charge for 3 balls to break even? What about for 5 balls?