MATH 7: HOMEWORK 8
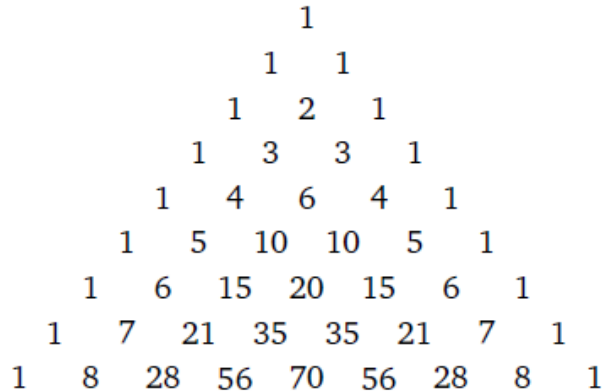**PASCAL TRIANGLE. APPLICATIONS.**
November 10, 2024

1. **Pascal's triangle and the binomial coefficients.**

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 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
\end{array}
$$

Every entry in it is obtained as the sum of two entries above it. The k-th entry in the n-th line is denoted by $\binom{n}{k}$, or by nC_k. Note that both n and k are counted from 0, not from 1:
for example, $\binom{6}{2} = 15$, $\binom{3}{0} = 1$. Now let us think about other applications of these numbers.

2. **Example applications of the binomial coefficients**
   a) Chessboard paths
   In the previous handout we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$\binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k units to the right.

For example, the number of paths that go to the upper right corner of a 6 × 6 board is equal to $\binom{10}{5}$, as *each* such path must have 5 steps to the right and 5 steps up, taken in any order. This means that we have total of n = 10 steps made of k = 5 steps up and the rest, 10 - 5 = 5 steps to the right. Or, form 10 possible moves we pick 5 to be up.

   b) Words with Rs and Us or 1s and 0s
Each of these paths on the board going up and to the right can be written as a sequence of steps; let R be a step to the right, and U a step up. For example, a possible path is: RRRRRUUUUU will go five steps to the right and five steps up, eventually ending in the upper right corner of a 6 × 6 board. Another possible sequence is RRUUURRRUU. There is a correspondence between paths of length **n** and strings of length **n** consisting of **Rs** and **Us** only. Now let us switch Rs to 0s and Us to 1s. We already know that $\binom{n}{k}$ = is the number of paths going k units up and n − k units to the right. This corresponds to words of length n with k Us and n − k Rs, which is the same as the number of strings of length n with k 1s and n − k Us. We have the following result:

$\binom{n}{k}$ = the number of words with length n that can be written using k ones and n − k zeroes or k Us and n-k Rs.

   c) Combinations: *the choice of **k** things from a set of **n** things without repetition ("replacement") and where the order does not matter.*
Consider now a set on n elements; let's number them from 1 to n. Then, for each string of length n with 0s and 1s, we can select those elements that correspond to 1s and omit those elements that correspond to 0s. In this way, we will get a subset of size k. This is another property of the binomial coefficients:

$\binom{n}{k}$ = the number of ways to choose k items out of n (order doesn't matter)

3. Summary: binomial coefficients represent

$nC_k = \binom{n}{k}$ = the number of paths on the chessboard going k units up and n − k to the right

= the number of words that can be written using k ones and n − k zeroes

= the number of ways to choose k items out of n (order doesn't matter)

Homework problems

Note: In the problems below, you can give your answer as a binomial coefficient **without calculating it**. If you want to calculate it, use the Pascal's triangle to find the value of $\binom{n}{k}$, where k is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. How many "words" of length 5 one can write using only letters U and R, namely 3 U's and 2 R's? What if you have 5 U's and 3 R's? [Hint: each such "word" can describe a path on the chessboard, U for up and R for right. . .]

2. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?

3. If we toss a coin 10 times, what is the probability that all will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?

4. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub — and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will move:

    a) 10 steps north
    b) 10 steps south
    c) 4 steps north
    d) will come back to the starting position

5. If you have a group of 4 people, and you need to choose one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?

6. How many ways are there to select 5 students from a group of 12?

7. In a meeting of 25 people, each much shake hands with each other. How many handshakes are there altogether?

8. (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?

    (b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).