MATH 7 HOMEWORK 5: Arithmetic sequences
October 20, 2024

1. Arithmetic sequence (progression)

A sequence of numbers is an arithmetic sequence if the difference between consecutive elements is the same number. This number is called a common difference, d.

For example: 1, 5, 9, 13, 17, ... The difference here is d = 4.

Sequence elements (terms) are labeled according to their position in the sequence using a counter **n** as a subscript. The value of the n-th element in a sequence is labeled as $\mathbf{a_n}$. Then, the first term in the sequence has n = 1 and a value of $a_1 = 1$, the second element is $a_2 = 5$, and so on.

We could find any element of a sequence knowing the first element a_1 and the difference d.
For example, what is a_{100}?

$a_1 = 1$
$a_2 = a_1 + d = 1 + 4 = 5$
$a_3 = a_2 + d = a_1 + 2d = 1 + 2 \times 4 = 9$
$a_4 = a_3 + d = a_1 + 3d = 1 + 3 \times 4 = 13$
...
$\boldsymbol{a_n = a_1 + (n-1)d}$

So $a_{100} = a_1 + 99d = 1 + 99 \times 4 = 397$

2.Property of an arithmetic sequence

A property of an arithmetic sequence is that any term is the arithmetic mean of its neighbors.

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Proof:

$$a_n = a_{n-1} + d$$
$$a_n = a_{n+1} - d$$

Add the left and the right sides:

$$2a_n = (a_{n-1} + d) + (a_{n-1} - d)$$
$$2a_n = a_{n-1} + a_{n-1}$$

Dividing by 2:

$$a_n = \frac{a_{n-1} + a_{n-1}}{2}$$

Another property of arithmetic sequences is that we can find the common difference d if we know any 2 terms a_s and a_t

$$d = \frac{a_s - a_t}{s - t}$$

3. Sum of an arithmetic sequence

$$S = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}$$

Proof: we write the sum in 2 ways, in increasing order and in decreasing order:

$S = a_1 + a_2 + a_3 + \cdots + a_n$
$S = a_n + a_{n-1} + a_{n-2} + \cdots + a_1$

Adding up left and right sides:

$$2S = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \cdots$$

We notice that:

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \cdots$$

$$2S = (a_1 + a_n) \times n$$

$$S = \frac{(a_1 + a_n) \times n}{2}$$

4. Arithmetic sequences -summary

$$\boldsymbol{a_n = a_1 + (n-1)d}$$

$$\boldsymbol{a_n = \frac{a_{n-1} + a_{n+1}}{2}}$$

$$\boldsymbol{d = \frac{a_s - a_t}{s - t}}$$

$$\boldsymbol{S = \frac{(a_1 + a_n) \times n}{2}}$$

Instructions: Please always write solutions on a ***separate sheet of paper***. Solutions should include explanations of **how you arrived at this answer.**

1. Write the first 5 terms of an arithmetic sequence if $a_1 = 7$ and $d = 2$.

2. What are the first two terms for the sequence $a_1, a_2, -9, -2, 5, \ldots$?

3. $a_{10} = 131$ and $d = 12$. What is a_1?

4. $a_5 = 27$ and $a_{27} = 60$. Find the first term a_1 and the common difference d.

5. Find the common difference d in an arithmetic sequence if the 9-th term is 18 and the 11-th term is 44.

6. In the arithmetic progression 5, 17, 29, 41, . . . what term has a value of 497?

7. Find the sum of the first 10 terms for the series: 4, 7, 10, 13, . . .

8. Find the sum of the first 1000 odd numbers.

9. Find the sum 2 + 4 + · · · + 2018.

10. In a given arithmetic progression, the first term is 6, and the 87-th term is 178. Find the common difference of this arithmetic progression and give the value of the first five terms.

11. The 3-rd term of the arithmetic progression is equal to 1. The 10-th term of it is three times as much as the 6-th term. Find the first term and the common difference. (**Hint:** Use the formula for the n-th term of the progression and write what is given in the problem using this formula.)

12. There are 25 trees at equal distances of 5 meters in a line with a well, the distance of the well from the nearest tree being 10 meters. A gardener waters all trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover to water all the trees.

13. * An arithmetic progression has first term $a_1 = a$ and common difference $d = -1$. The sum of the first n terms is equal to the sum of the first $3n$ terms. Express a in terms of n.

14. * The sum of the first 20 terms of an arithmetic progression is 200, and the sum of the next 20 terms is -200. Find the sum of the first hundred terms of the progression.