

## Homework 8: Pascal's triangle, applications

HW8 is Due Dec 8.

### 1. Pascal's triangle and the binomial coefficients.

				1																
				1		1														
				1		2		1												
				1		3		3		1										
				1		4		6		4		1								
				1		5		10		10		5		1						
				1		6		15		20		15		6		1				
				1		7		21		35		35		21		7		1		
				1		8		28		56		70		56		28		8		1

Every entry in it is obtained as the sum of two entries above it. The  $k$ -th entry in the  $n$ -th line is denoted by  $\binom{n}{k}$ , or by  $nC_k$ . Note that both  $n$  and  $k$  are counted from 0, not from 1: for example,  $\binom{6}{2} = 15$ ,  $\binom{3}{0} = 1$ . Now let us think about other applications of these numbers.

### 2. Example applications of the binomial coefficients

#### a) Chessboard paths

In the previous handout we saw that these numbers appear in a problem about counting paths from the lower left corner of the board to the upper right corner. We observed the following:

$\binom{n}{k}$  = the number of paths on the chessboard going  $k$  units up and  $n - k$  units to the right.

For example, the number of paths that go to the upper right corner of a  $6 \times 6$  board is equal to  $\binom{10}{5}$ , as each such path must have 5 steps to the right and 5 steps up, taken in any order. This means that we have total of  $n = 10$  steps made of  $k = 5$  steps up and the rest,  $10 - 5 = 5$  steps to the right. Or, from 10 possible moves we pick 5 to be up.

#### b) Words with Rs and Us or 1s and 0s

Each of these paths on the board going up and to the right can be written as a sequence of steps; let R be a step to the right, and U a step up. For example, a possible path is: RRRRRUUUUU will go five steps to the right and five steps up, eventually ending in the upper right corner of a  $6 \times 6$  board. Another possible sequence is RRUUURRRUU. There is a correspondence between paths of length  $n$  and strings of length  $n$  consisting of **Rs** and **Us** only. Now let us switch Rs to 0s and Us to 1s. We already know that  $\binom{n}{k}$  = is the number of paths going  $k$  units up and  $n - k$  units to the right. This corresponds to words of length  $n$  with  $k$  Us and  $n - k$  Rs, which is the same as the number of strings of length  $n$  with  $k$  1s and  $n - k$  0s. We have the following result:

$\binom{n}{k}$  = the number of words with length  $n$  that can be written using  $k$  ones and  $n - k$  zeroes or  $k$  Us and  $n - k$  Rs.

#### c) Combinations: the choice of $k$ things from a set of $n$ things without repetition ("replacement") and where the order does not matter.

Consider now a set on  $n$  elements; let's number them from 1 to  $n$ . Then, for each string of length  $n$  with 0s and 1s, we can select those elements that correspond to 1s and omit those elements that correspond to 0s. In this way, we will get a subset of size  $k$ . This is another property of the binomial coefficients:

$\binom{n}{k}$  = the number of ways to choose  $k$  items out of  $n$  (order doesn't matter)

### 3. Summary: binomial coefficients represent

$nC_k = \binom{n}{k}$  = the number of paths on the chessboard going k units up and n – k to the right  
= the number of words that can be written using k ones and n – k zeroes  
= the number of ways to choose k items out of n (order doesn't matter)

#### Homework problems

**Instructions:** Please always write solutions on a **separate sheet of paper**. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

**Note:** In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use the Pascal's triangle to find the value of  $\binom{n}{k}$ , where k is the k-th element in the n-th row of the Pascal triangle, counting from 0.

1. Which of the numbers in Pascal triangle are even? Can you *guess the pattern*, and then carefully explain why it works? (no formulas)
2. What is the sum of all entries in the n-th row of Pascal triangle? Try computing the first several answers and then guess the general formula.
3. How many "words" of length 5 one can write using only letters U and R, namely 3 U's and 2 R's? What if you have 5 U's and 3 R's? [Hint: each such "word" can describe a path on the chessboard, U for up and R for right. . . ]
4. How many sequences of 0 and 1 of length 10 are there? sequences of length 10 containing exactly 4 ones? exactly 6 ones?
5. If we toss a coin 10 times, what is the probability that all will be heads? that there will be exactly one tails? 2 tails? exactly 5 tails?
6. If you have a group of 4 people, and you need to choose one to go to a competition, how many ways are there to do it? if you need to choose 2? if you need to choose 3?
7. How many ways are there to select 5 students from a group of 12?
8. In a meeting of 25 people, each much shake hands with each other. How many handshakes are there altogether?
9. (a) An artist has 12 paintings. He needs to choose 4 paintings to include in an art show. How many ways are there of doing this?  
(b) The same artist now needs to choose 4 paintings to include in a catalog. How many ways are there to do this? (In the catalog, unlike the show, the order matters).