MATH 6: HANDOUT 11
GEOMETRY: RULER AND COMPASS CONSTRUCTIONS

MONTY HALL PROBLEM

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

Is it to your advantage to switch your choice?

Answer: yes!

Let us see what happens if you have chosen door No. 1. Then there are the following possibilities:

Car is behind door..	Probability	It is better to
1	1/3	stay
2	1/3	switch
3	1/3	switch

Thus, switching wins with probability 2/3.

Here is one more problem of a similar sort.

EXCHANGE PARADOX (OR TWO-ENVELOPE PROBLEM)

You are given two indistinguishable envelopes, each of which contains some amount of money. It is known that one envelope contains 10 times as much as the other. You pick one envelope at random; it is opened and you see that it contains \$50. Now you are given a choice: do you want to keep this money — or do you want to take the other envelope instead?

CONSTRUCTIONS WITH RULER AND COMPASS

For the next couple of classes we will be mostly interested in doing the geometric constructions with ruler and compass. Note that the ruler can only be used for drawing straight lines through two points, not for measuring distances!

When doing these problems, we need:

- Give a recipe for constructing the required figure using only ruler and compass
- **Explain why our recipe does give the correct answer**

For the first part, our recipe can use only the following operations:

- Draw a line through two given points
- Draw a circle with center at a given point and given radius
- Find and label on the figure intersection points of already constructed lines and circles.

For the second part, we will frequently use the results below.

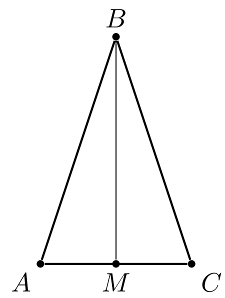
CONGRUENCE TESTS FOR TRIANGLES

Recall that by definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalities. However, it turns out that in fact, we can do with fewer checks.

Axiom 1. *(SSS Rule). If $AB = A'B', BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 2. *(ASA Rule). If $\angle A = \angle A', \angle B = \angle B'$ and $AB = A'B'$ then $\triangle ABC \cong \triangle A'B'C'$.*

Axiom 3. *(SAS Rule). If $AB = A'B', AC = A'C'$ and $\angle A = \angle A'$ then $\triangle ABC \cong \triangle A'B'C'$.*



Isosceles Triangle

Recall that the triangle $\triangle ABC$ is called isosceles if $AB = BC$.

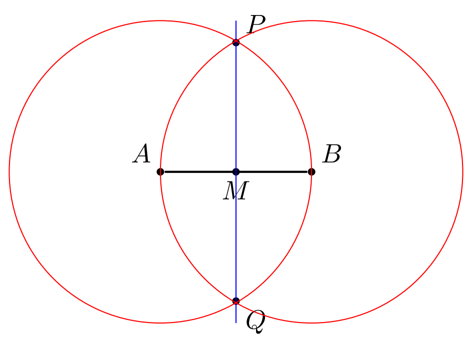
Theorem 1. *Properties of an isosceles triangle:*

1. *In an isosceles triangle, base angles are equal:* $\angle A = \angle C$.
2. *In an isosceles triangle, let M be the midpoint of the base AC. Then line BM is also the bisector of angle B and the altitude: BM is perpendicular to AC.*

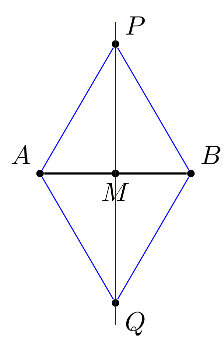
Example: Finding the midpoint of the line segment

Problem: given two points A, B, construct the midpoint M of the segment AB.
Construction:

1. Draw a circle with center at A and radius AB
2. Draw a circle with center at B and radius AB
3. Mark the two intersection points of these circles by P, Q
4. Draw line through points P, Q
5. Mark the intersection point of line PQ with line AB by M. This is the midpoint.



Analysis: This is a two-step argument. In this figure, triangles $\triangle APQ$ and $\triangle BPQ$ are congruent (*why?*), so the corresponding angles are equal:



From this, we can see that $\triangle APM \cong \triangle BPM$, so $AM = BM$.

Homework

When in the problems below we say that some *lengths are given*, assume that there is an interval of a given length already drawn on the paper.

1. Explain why in the construction above, the line PQ will in fact be a perpendicular to AB.
2. Given a line l and a point A on l, construct a perpendicular to l through A.
3. Given a line l and a point P outside of l, construct a perpendicular to l through P.
4. Given an angle AOB, construct the angle bisector (i.e., a ray OM such that $\angle AOM \cong \angle BOM$).
5. Given length a, construct an equilateral triangle with side a.
6. Given length a, construct a regular hexagon with side a.
7. Given three lengths a, b, c, construct a triangle with sides a, b, c.
8. Construct an isosceles triangle, given a base b and altitude h.
9. Construct a right triangle, given a hypotenuse h and one of the legs a.
10. You have two fuses (specially treated cords, which burn slowly and reliably). Each of them would burn completely for one minute if lighted from one end. Using this, can you measure the time of 30 seconds? of 45 seconds? Note: some parts of the fuses may burn faster than others — so you can not just measure half of the fuse and say that it will burn for exactly 30 seconds.