MATH 6: HANDOUT 9
PERMUTATIONS AND FACTORIALS

One common problem in Math is counting in how many different ways things can be ordered or arranged. In cases in which we do not have repeated objects and the order in which we arrange them matters, we call this a **permutation**.

Imagine, for example, that we are in charge of a natural reserve with four habitats and we have four different animals that we have to assign to each habitat: lion, unicorn, koala, sloth. We want to count in how many different ways we could arrange them.

🦁 🦄	________	________	________	________
🐨 🦥	Habitat 1	Habitat 2	Habitat 3	Habitat 4

An easy way to think about this is to first consider how many different possibilities you have for assigning an animal in the first habitat. Since there are four animals, we have four possibilities:

🦁 🦄	4 possibilities	________	________	________
🐨 🦥	Habitat 1	Habitat 2	Habitat 3	Habitat 4

Suppose that we do not want the sloth to walk so much, so we assign it the first habitat. Then, we need to fill the second habitat, but now we will only have three possibilities, since there are three animals left:

🦁 🦄	🦥	3 possibilities	________	________
🐨	Habitat 1	Habitat 2	Habitat 3	Habitat 4

As we keep making choices, the number of possible choices for the next case will decrease by one.

🦄	🦥	🦁	2 possibilities	________
🐨	Habitat 1	Habitat 2	Habitat 3	Habitat 4

Finally, we only have one possibility for the missing animal:

	🦥	🦁	🦄	1 possibility
🐨	Habitat 1	Habitat 2	Habitat 3	Habitat 4

To find the total number of possibilities in which we can arrange the animals, we **multiply** the different possibilities that we had for each habitat:

$$\underline{\qquad 4 \qquad} \times \underline{\qquad 3 \qquad} \times \underline{\qquad 2 \qquad} \times \underline{\qquad 1 \qquad} = 24$$

Imagine that instead of having 4 animals and 4 habitats, we have n animals and n habitats (where n could be any number). In this case, we have n possibilities for the first choice, $n-1$ for the second, and so on, leaving a total of

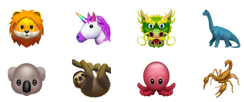
$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \text{ possibilities}$$

This multiplication is so common, that there was a function defined to represent it called the **factorial**. This is written as $n!$ and it is defined as:

$$n! \equiv n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

By convention, we define $0! = 1$.

Imagine that now we have 8 animals, but only three habitats. In this case, we will have the following:



$\underline{\text{8 possibilities}}$	$\underline{\text{7 possibilities}}$	$\underline{\text{6 possibilities}}$
Habitat 1	Habitat 2	Habitat 3

In this case, we will have $8 \times 7 \times 6 = 336$ possibilites.

In general, if we are choosing k objects from a collection of n so that a) order matters and b) no repetitions are allowed, then this is referred to as a *permutation* of k objects from the collection of n, the number of ways to make such a selection of permutations is called ${}_nP_k$, and

$${}_nP_k = n \times (n-1) \times \cdots \times (n-k+1) \quad (k \text{ factors})$$

Using factorials, we can give a simpler formula for ${}_kP_n$:

$${}_nP_k = \frac{n!}{(n-k)!}$$

For example:

$${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{6!}{2!}$$

Homework

1. In this problem, you have to express your answer as a simplified exponent (you do not have to compute numerically the expressions that you find). Simplify the expressions below using the power laws: $(a \times b)^n = a^n \times b^n$, $(a^n)^m = a^{n \times m}$, $a^n a^m = a^{n+m}$, $a^n/a^m = a^{n-m}$, $a^{-n} = 1/a^n$ and $a^0 = 1$.

(a) $\frac{2^5 4^4}{2^7}$ (b) $6^5 \times 3^{-4}$ (c) $\frac{5^{-2}}{5^{-4}}$

2. You have 4 shirts and 4 ties colored red, yellow, blue and green. How many shirt and tie combinations are there if you refuse to wear a shirt and a tie of the same color?

3. A sly elementary school teacher decides to play favorites without telling anyone. If they have 15 students in their class, in how many ways can they choose a favorite student, a second favorite student, and a third favorite student?

4. Compute $\frac{5!}{4!}$, $5! - 4!$, ${}_3P_2$, ${}_3P_3$

5. (a) How many ways are there to draw 3 cards from a 52-card deck? (Order matters: drawing first king of spades, then queen of hearts is different from drawing them in opposite order).
(b) How many ways are there to draw 3 cards from a 52-card deck if after each drawing we record the card we got, then return the card to the deck and reshuffle the deck? (As before, order matters.)

6. Polly is talking parrot who speaks in 3-word sentences. A Polly's sentence always starts with a pronoun, which is followed by a verb, and then by a noun. Polly knows:

- 2 pronouns: I and WE
- 3 verbs: LOVE, WANT, and COOK,
- 4 nouns: FOOD, CRACKER, FRIEND, and SHMOLLAR

Polly's friend Dolly the parrot can talk as well. A Dolly's sentence always starts with an adjective, which is followed by a noun, and then by a verb:

- 3 adjectives: HAPPY, HUNGRY, and LONELY,
- 2 nouns: PARROT and CROCODILE,
- 3 verbs: SINGS, CRIES, and WORKS.

(a) How many different phrases can Polly the parrot say?
(b) How many different sentences can Dolly the parrot say?
(c) Polly and Dolly are creating a two-phrase story. Each parrot contributes a sentence. How many different stories can they come up with?

7. Going back to the example with the animal habitats, how many different possible arrangements can you have if you have 6 animals and 15 habitats?

8. [**Discussed in class already.**] A group of 6 club members always dine at the same table in the club; there are exactly 6 chairs at the table. They decided that each day, they want to seat in a different order. Can they keep this for a year? Two years?

9. How many ways are there to seat 15 students in a classroom which has 15 chairs? If the room has 25 chairs?

10. 10 people must form a circle for some dance. In how many ways can they do this?