Math 5e, Homework 21
due March 12

Instructions: Some of the problems we solved in class, and some are new. Please try to solve all problems, do your best, and show your work. Write on separate sheets of paper, not between the lines of this handout!

Introduction to Probability

In general, the probability of obtaining one outcome from a certain collection of A possible outcomes, is given by

$$P(A) = \frac{Number\ of\ outcomes\ giving\ A}{total\ number\ of\ possible\ outcomes}$$

Product Rule – what we did in class

Question: we roll two dice. What is the probability of rolling a 5 and a 6?

Answer: There are two ways of getting a 5 and a 6: as pair (5, 6) (5 on die number 1, 6 on die number 2) or as (6, 5) (6 on die number 1, 5 on die number 2). Thus, the answer is $\frac{2}{36} = \frac{1}{18}$.

Question: if one tosses a coin 10 times, what is the probability that all will be heads?

Answer: $\frac{1}{2^{10}}$ (using a calculator, one can compute that this is $1/1024 \approx 0.001$, or 1/10 of 1%).

Question: if one tosses a coin 10 times, what is the probability that all will be tails?

Answer: The same as all heads.

Question: if one tosses a coin 10 times, what is the probability that *at least* one will be heads?

Answer: Unfortunately, too many combinations give at least one head. In fact, it is easier to say which combinations do not give at least one head: there is exactly one such combination, all tails, which has a probability of $1\frac{1}{2^{10}} = \frac{1}{1024}$. The remaining combinations will give at least one head; thus, the probability of obtaining at least one head is $1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999$

Percentages and Fractions

It is also common to express probabilities as percentages: by definition, $1\% = \frac{1}{100} = 0.01$ so $x\% = \frac{x}{100}$. For example, $3\% = \frac{3}{100} = 0.03$, and $1.5\% = \frac{1.5}{100} = 0.015$. This conversion is necessary when you multiply probabilities, as the following example shows:

> **Question.** The probability of winning in a certain game is p = 5%. What is the probability of winning two times in a row?
>
> **Answer.** According to the product rule, it is $p \times p = p^2$. However, the answer $5\% \times 5\% = 25\%$ is wrong. Correct answer is $\frac{5}{100} \times \frac{5}{100} = 0.0025$.

To convert from decimals to percent, multiply by 100: $p = (p \times 100)\%$. For example, $\frac{1}{5} = 0.2 = (0.2 \times 100)\% = 20\%$

Formulas for fast multiplication

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$(a - b)(a + b) = a^2 - b^2$$

Homework problems

1. Simplify the fractions using factorization and the formulas for fast multiplication. Remember, $(1 = 1^2)$:

   (a) Examle: $\dfrac{169-a^2}{a^2-26a+13} = \dfrac{13^2-a^2}{(a-13)^2} = \dfrac{(13-a)(13+a)}{(a-13)(a-13)} = \dfrac{-\cancel{(a-13)}(13+a)}{\cancel{(a-13)}(a-13)} =$

   $\dfrac{-(13+a)}{a-13} = \dfrac{13+a}{13-a}$

   (b) $\dfrac{x^2-2x+1}{x^2-1} =$

   (c) $\dfrac{x^3-x}{x^2+x} =$

2. If we roll two dice, what is the probability that the product of two numbers is a multiple of 3? or a multiple of 5?

3. Recall that a roulette wheel has 37 slots: 0 through 36. Among slots 1–36, half are red, and the other half are black (zero has no color). What is the probability of obtaining
   (a) red (on a single run of roulette)
   (b) red, then black, then 0 (on 3 successive runs)
   (c) red 15 times in a row?
   (d) blue (on a single run of roulette)
   (e) this sequence of colors: RRRBRBRBBRBBRBR (also of length 15)?

4. A hunter is shooting ducks. The probability of hitting a duck with one shot is $p = \frac{1}{3}$.
   (a) What is the probability of missing the duck (with one shot)?
   (b) He makes 5 shots. What is the probability that he misses all five times?
   (c) What is the probability that out of 5 shots, he will hit at least once? Will this probability double if he makes 10 shots? (You can use the calculator to compute the answers)
   (d) What is the probability that out of 5 shots, he will hit exactly once? Will this probability double if he makes 10 shots?

5. A license plate consists of 3 letters followed by three digits. How many possible license plates are there?

6. In one kind of lottery, they put balls with numbers 1 through 100 in a bag and then draw six balls at random (the drawn ball is put aside and not returned to the bag). To win the lottery, one needs to guess all six numbers in correct order. What is the probability of this?