Math 5b. Classwork23.

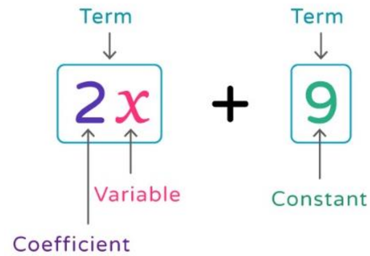
Algebraic expressions.

Algebraic expression, or variable expression, is a mathematical expression consisting of two main parts, variables and constants, joined together using mathematical operators addition, subtraction, multiplication, division, and exponentiation.

Here are a few examples of algebraic expressions:

$$2x^5; \qquad 2+7y; \qquad 3a-5b; \qquad \frac{2+x}{2-x} = (2+x):(2-x), x \neq 0$$

In the expression $3a-5b$ a and b are variables, 3 and 5 are coefficients, $3a$ and $5b$ are both terms.



In the expression $2+7y$, y is variables, 7 is coefficients, 2 is a constant and both 2 and $7y$ are both terms.

We can combine like terms and cannot combine unlike terms:

$$2x^5 + 4x^5 = 6x^5 \qquad but \qquad 2x^5 + 4x^3 \neq 6x^?$$

First equality is always the true for any x, second expression can't be defined.

(For the second expression we can find two values for x such that the left side of the expression will be equal to the right side, and the power of x on the right side is irrelevant. If $x = 0 \; or \; 1$:

$2 \cdot 1^5 + 4 \cdot 1^3 = 6 \cdot 1^? \qquad 2 \cdot 0^5 + 4 \cdot 0^3 = 6 \cdot 0^?$)

We can work with the algebraic expressions in a very similar way as we work with numbers. We can add them, subtract them, multiply them and divide them (multiply by the invers expression).

For example:

Add the expressions: $4x - 3y$ and $5x + 7y - 4$

$(4x-3y) + (5x+7y-4) = 4x - 3y + 5x + 7y - 4 = 9x + 4y - 4$

Subtract the expressions: $5a - 7b^2$ and $7a + 7b - b^2 + 3$

$(5a - 7b^2) - (7a + 7b - b^2 + 3) = 5a - 7b^2 - 7a - 7b + b^2 - 3 = -2a - 6b^2 - 7b - 3$

Multiply $6z^2$ and $7b - b^2$:

$$
\begin{aligned}
6z^2 \cdot (7b - b^2)& \\
&= \underbrace{(7b - b^2) + (7b - b^2) + .. + (7b - b^2)}_{6z^2 times} = \underbrace{7b - b^2 + 7b - b^2 + \cdots + 7b - b^2}_{6z^2 times} \\
&= \underbrace{7b + 7b + \cdots + 7b}_{6z^2 times} + \underbrace{(-b^2) + (-b^2) + \cdots + (-b^2)}_{6z^2 times} = 6z^2 \cdot 7b + 6z^2 \cdot (-b^2) \\
&= 42z^2b - 6z^2b^2
\end{aligned}
$$

Each term of the expression $(7b - b^2)$ is multiplied by $6z^2$ (a single term expression).

Multiply $3p^2 + a$ and $2b - a + 3$:

$$
\begin{aligned}
(3p^2 + a)(2b - a + 3) &= 3p^2 \cdot 2b - 3p^2 \cdot a + 3p^2 \cdot 3 + a \cdot 2b - a \cdot a + 3a \\
&= 6p^2b - 3p^2a + 9p^2 + 2ab - a^2 + 3a
\end{aligned}
$$

Irrational numbers

Rational number is a number which can be represented as a ratio of two integers:

$$a = \frac{p}{q}; \qquad p \in Z, and\ q \in N, \qquad (Z = \{\pm \cdots, \pm 1, 0\}, N = \{1, 2, \dots\})$$

Rational numbers can be represented as infinite periodical decimals (in the case of denominators containing only powers of 2 and 5 the periodical bloc of such decimal is 0).

Numbers, which can't be express as a ratio (fraction) $\frac{p}{q}$ for any integers p and q are irrational numbers. Their decimal expansion is not finite, and not periodical.

Examples:

0.01001000100001000001…

0.123456789101112131415161718192021…

What side the square with the area of a m^2 does have? To solve this problem, we have to find the number, which gives us a as its square. In other words, we have to solve the equation

$$x^2 = a$$

This equation can be solved (has a real number solution) only if a is nonnegative (($a \geq 0$) number. It can be seen very easily;

If $x = 0, \quad x \cdot x = x^2 = a = 0,$

If $x > 0, \quad x \cdot x = x^2 = a > 0,$

If $x < 0, \quad x \cdot x = x^2 = a > 0,$

We can see that the square of any real number is a nonnegative number, or there is no such real number that has negative square.

Square root of a (real nonnegative) number a is a number, square of which is equal to a.

There are only 2 square roots from any positive number, they are equal by absolute value, but have opposite signs. The square root from 0 is 0, there is no any real square root from negative real number.

Examples:

1. Find square roots of 16: 4 and (-4), $4^2 = (-4)^2 = 16$
2. Numbers $\frac{1}{7}$ and $\left(-\frac{1}{7}\right)$ are square roots of $\frac{1}{49}$, because $\frac{1}{7} \cdot \frac{1}{7} = \left(-\frac{1}{7}\right) \cdot \left(-\frac{1}{7}\right) = \frac{1}{49}$
3. Numbers $\frac{5}{3}$ and $\left(-\frac{5}{3}\right)$ are square roots of $\frac{25}{9}$, because $\left(\frac{5}{3}\right)^2 = \frac{5}{3} \cdot \frac{5}{3} = \left(-\frac{5}{3}\right)^2 = \left(-\frac{5}{3}\right) \cdot \left(-\frac{5}{3}\right) = \frac{25}{9}$

Arithmetic (principal) square root of a (real nonnegative) number a is a nonnegative number, square of which is equal to a.

There is a special sign for the arithmetic square root of a number a: $\sqrt{a}$.
Examples;

1. $\sqrt{25} = 5$, it means that arithmetic square root of 25 is 5, as a nonnegative number, square of which is 25. Square roots of 25 are 5 and (-5), or $\pm\sqrt{25} = \pm 5$
2. Square roots of 121 are 11 and (-11), or $\pm\sqrt{121} = \pm 11$
3. Square roots of 2 are $\pm\sqrt{2}$.
4. A few more:
   $\sqrt{0} = 0$; $\sqrt{1} = 1$; $\sqrt{4} = 2$; $\sqrt{9} = 3$; $\sqrt{16} = 4$;
   $\sqrt{25} = 5$; $\sqrt{\frac{1}{64}} = \frac{1}{8}$; $\sqrt{\frac{36}{25}} = \frac{6}{5}$

Base on the definition of arithmetic square root we can right

$$\left(\sqrt{a}\right)^2 = a$$

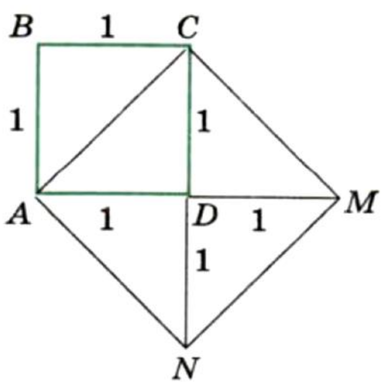
To keep our system of exponent properties consistent let's try to substitute $\sqrt{a} = a^k$. Therefore,

$$\left(\sqrt{a}\right)^2 = (a^k)^2 = a^1$$

But we know that

$$(a^k)^2 = a^{2k} = a^1 \quad \Rightarrow \quad 2k = 1,\ \ k = \frac{1}{2}$$

To solve equation $x^2 = 23$ we have to find two sq. root of 23. $x = \pm\sqrt{23}$. 23 is not a perfect square as 4, 9, 16, 25, 36 …



The length of the segment [AC] is $\sqrt{2}$ (from Pythagorean theorem). The area of the square ACMN is twice the area of the square ABCD. Let assume that the $\sqrt{2}$ is a rational number, so it can be represented as a ratio $\frac{p}{q}$, where $\frac{p}{q}$ is nonreducible fraction.

$$\left(\frac{p}{q}\right)^2 = 2 = \frac{p^2}{q^2}$$

Or $p^2 = 2q^2$, therefore p^2 is an even number, and p itself is an even number, and can be represented as $p = 2p_1$., consequently

$$p^2 = (2p_1)^2 = 4{p_1}^2 = 2q^2$$

$2{p_1}^2 = q^2 \Rightarrow q$ also is an even number and can be written as $q = 2q_1$.

$$\frac{p}{q} = \frac{2p_1}{2q_1}$$

therefore fraction $\frac{p}{q}$ can be reduced, which is contradict the assumption. We proved that the $\sqrt{2}$ isn't a rational number by contradiction.

$\sqrt{2}$ is an irrational number, therefore its decimal representation is an infinite nonperiodically decimal:

$$\sqrt{2} = a_0.a_1a_2 \ldots$$

$$1 < 2 < 4; \quad \Rightarrow \quad \sqrt{1} < \sqrt{2} < \sqrt{4}; \qquad 1 < \sqrt{2} < 2; \qquad \Rightarrow \quad a_0 = 1$$

To find a_1 let's consider numbers $1.0,\ \ 1.1,\ \ 1.2,\ \ 1.3$ ….

$$1.0^2 = 1; \quad 1.1^2 = 1.21; \quad 1.2^2 = 1.44$$

$$1.3^2 = 1.69; \quad 1.4^2 = 1.96; \quad 1.5^2 = 2.25$$

Therefore:

$$1.4^2 < 2 < 1.5^2, \qquad 1.4 < \sqrt{2} < 1.5$$

$$\sqrt{2} = 1.4a_2a_3 \ldots$$

To find the next digit,

$$1.40^2 = 1.96; \quad 1.41^2 = 1.9881; \quad 1.42^2 = 1.2.0164$$

$$\sqrt{2} = 1.41a_3 \ldots; \quad \sqrt{2} = 1.4142135 \ldots$$

Exercises.

1. Simplify the following expressions (combine like terms):

$a.$ $7a + (2a + 3b);$ $b.$ $9x + (2y - 5x);$
$c.$ $(5x + 7a) + 4x;$ $d.$ $(5x - 7a) + 5a;$
$e.$ $(3x - 6y) - 4y;$ $f.$ $(2a + 5b) - 7b;$
$g.$ $3m - (5n + 2m);$ $h.$ $6p - (5p - 3a);$

2.

$a.$ $(x^2 + 4x) + (x^2 - x + 1) - (x^2 - x);$
$b.$ $(a^5 + 5a^2 + 3a - a) - (a^3 - 3a^2 + a);$
$c.$ $(x^2 - 3x + 2) - (-2x - 3);$
$d.$ $(abc + 1) + (-1 - abc);$

.

3. Simplify the following expressions (rewrite the expressions without parenthesis, combine like terms);

Example:

$$(2x + 3) \cdot (x + 7) = 2xx + 2x \cdot 7 + 3x + 3 \cdot 7 = 2x^2 + 10x + 21$$

$(x + 5)(x + y + 3);$

$(k - 1 + d)(k - d);$

$\frac{2}{3} + 2x\left(\frac{1}{2} - \frac{1}{3}y\right) - x - \frac{1}{3}(2 - 2xy);$

$2x^2(x + y) - 3x^2(x - y);$

4. Evaluate:

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}$$

5. Prove that the value of the following expressions is a rational number.

a. $(\sqrt{2} - 1)(\sqrt{2} + 1)$

b. $\left(\sqrt{5}-\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)$

c. $\left(\sqrt{2}+1\right)^2+\left(\sqrt{2}-1\right)^2$

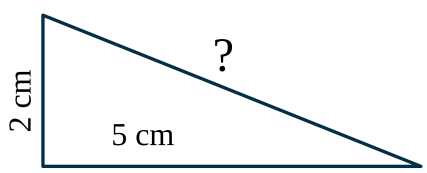
d. $\left(\sqrt{7}-1\right)^2+\left(\sqrt{7}+1\right)^2$

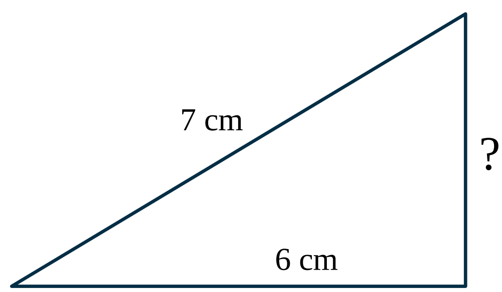
e. $\left(\sqrt{7}-2\right)^2+4\sqrt{7}$

6. Without using calculator compare:

$3 \ \dots \ \sqrt{11}$	$11 \ \dots \ \sqrt{110}$	$22 \ \dots \ \sqrt{484}$
$5 \ \dots \ \sqrt{20}$	$17 \ \dots \ \sqrt{299}$	$35 \ \dots \ \sqrt{1215}$

7. Find the missing length of the side of right triangles below:





8. Evaluate:

$a.\ 5\cdot\sqrt{4}\cdot 3;$ $\quad b.\ 2\cdot\sqrt{9}+3\cdot\sqrt{16}$

$c.\ \sqrt{13-3\cdot 3};$ $\quad d.\ \sqrt{7^2-26:2}$

$e.\ \frac{1}{2}\sqrt{5^2+22:2};$ $\quad f.\ 3\sqrt{0.64}-5\cdot\sqrt{1.21}$