Math 5b. Classwork 3.

1. $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any $a \neq 0$
6. $(a \cdot b)^n = a^n \cdot b^n$

Positive and negative numbers:

- A positive number raised into any power will result a positive number.
- A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

If a number a in a power n is divided by the same number in a power m,

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m}$$

So, we can say that

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a}; \qquad a^{-n} = \frac{1}{a^n};$$

Let's see how our decimal system of writing numbers works when we use the concept of exponent:

$$3456 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 = 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6$$

The value of a place of a digit is defined by a power of 10 multiplied by the digit. Very large numbers can be written using this system, as well as very small numbers.

$$0.3 = \frac{1}{10} \cdot 3 = \frac{1}{10^1} = 10^{-1} \cdot 3$$

$$24.345 = 10^1 \cdot 2 + 10^0 \cdot 4 + 10^{-1} \cdot 3 + 10^{-2} \cdot 4 + 10^{-3} \cdot 5$$

Scientists work with very large and very small things, from galaxies to viruses. They need to be able to write numbers, describing the object of interest, for example the distance between two galaxies or the diameter of a virus.

One of the most important numbers in the universe is the speed of light.

299 792 458 m / s. It’s very convenient to represent it as a decimal starting with units and multiplied by a power of 10.

$$299\ 792\ 458\ m\ per\ s = 2.99792458 \cdot 10^{8}\ m\ p\ s.$$

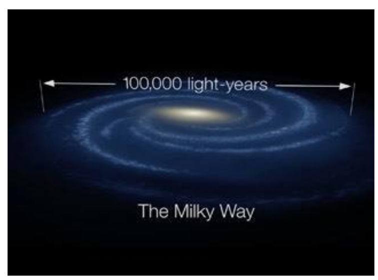
Let's convert the value to kilometers per hour. Each kilometer is 1000 meters, so we need to divide it by 1000:

$$3 \cdot \frac{10^{8}}{10^{3}} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^{5} km\ per\ s$$

In each hour there are 3600 seconds, or $3.6 \cdot 10^{3}$ seconds. To find out the speed of light in km per hour we now need to multiply the speed in seconds by $3.6 \cdot 10^{3}$

$$3 \cdot \frac{10^{8}}{10^{3}} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^{5} km\ per\ s = 3 \cdot 10^{5} \cdot 3.6 \cdot 10^{3} = 10.8 \cdot 10^{8}\ km\ p\ h$$

The Milky Way galaxy has a diameter of 105,700 light years, so the light will travel from one end to the other through its center in 105700 years.

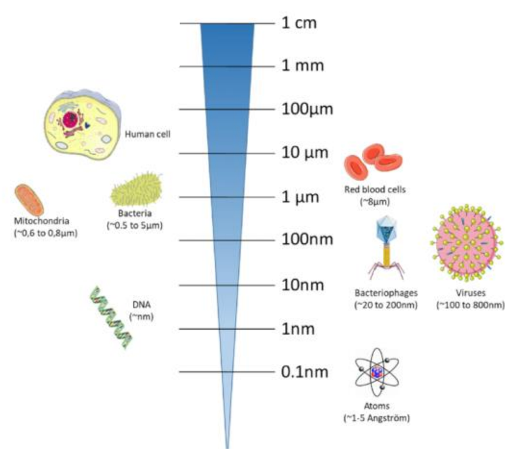


How far is one side from the other in the Milky Way in kilometers? 10.8·108$km\ p\ h$ ·105700 $years$.

How many hours in a year? $24 \cdot 365.25 = 8766 \approx 8.8 \cdot 10^{3}\ hours$

$$10.8 \cdot 10^{8}\ km\ p\ h\ \cdot 105700\ years \approx 1.08 \cdot 10^{9}\ km\ p\ h \cdot 8.8 \cdot 10^{3} \cdot 1.06 \cdot 10^{5}$$
$$\approx 10.07 \cdot 10^{17} km \approx 10^{18} km.$$

This way to write numbers is called scientific notation, it’s used a lot in science for describing various objects, big and small. The number should be represented as a product of one-digit whole part of a number with decimal part rounded up to two digit after the point and a power of 10. For example, the distance from the Earh to the Moon is about 238,855 miles (384,400 kilometers).



$$238855 = 2.38855 \cdot 10^{5} \approx 2.39 \cdot 10^{5}\ miles$$

There's a famous legend about the origin of chess that goes like this. When the inventor of the game showed it to the emperor of India, the emperor was so impressed by the new game, that he said to the man

"*Name your reward!*"

The man responded,

"*Oh emperor, my wishes are simple. I only wish for this. Give me one grain of rice for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before.*"

The emperor agreed, amazed that the man had asked for such a small reward - or so he thought. After a week, his treasurer came back and informed him that the reward would add up to an astronomical sum, far greater than all the rice that could conceivably be produced in many centuries!

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64} = 36893488147419103230$$

This number is huge! We can write it as $3.6893488147419103230 \cdot 10^{19} \approx 3.69 \cdot 10^{19}$

Let's take a look on the small things, like bacteria and viruses.

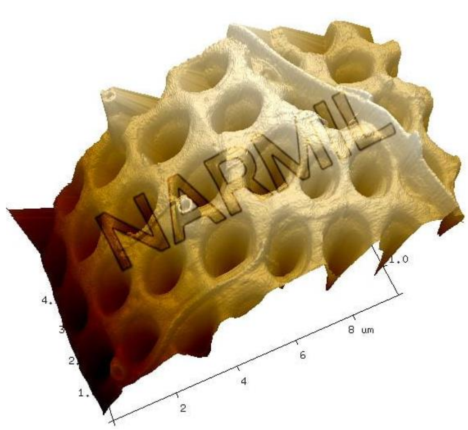
$1cm = 0.01m = \frac{1}{100}m = \frac{1}{10^2}m = 10^{-2}m$

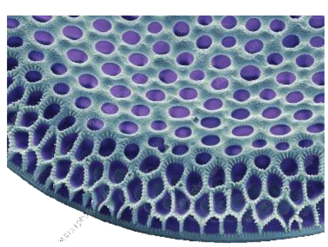
$1mm = 0.001m = \frac{1}{1000}m = \frac{1}{10^3}m = 10^{-3}m$

$1\mu m = 10^{-6}m, \quad 1nm = 10^{-9}m$

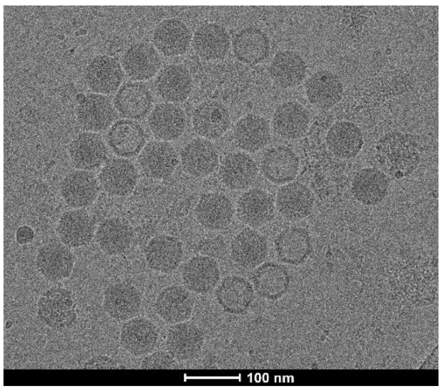
Bactria are between $0.5 - 1.5\ \mu m$

$0.5\mu m = 0.5 \cdot 10^{-6}m = 5 \cdot 10^{7}m$





Diatoms are aquatic organisms with transparent shells made of silica. These little guys come in all sorts of odd shapes and sizes. Their shells contain distinct patterns and perforations, allowing substances to pass in and out of the cell.



Prefix	Symbol for Prefix		Scientific Notation
exa	E	1 000 000 000 000 000 000	10^{18}
peta	P	1 000 000 000 000 000	10^{15}
tera	T	1 000 000 000 000	10^{12}
giga	G	1 000 000 000	10^{9}
mega	M	1 000 000	10^{6}
kilo	k	1 000	10^{3}
hecto	h	100	10^{2}
deka	da	10	10^{1}
----	--	1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	µ	0.000 001	10^{-6}
nano	n	0.000 000 001	10^{-9}
pico	p	0.000 000 000 001	10^{-12}
femto	f	0.000 000 000 000 001	10^{-15}
atto	a	0.000 000 000 000 000 001	10^{-18}

Exercises:

1. It is known that $a+1$ is divisible by 3. Prove that $4+7a$ is divisible by 3 as well.
2. You are offered a job, which lasts for **7 weeks**. You get to choose your salary.

   - **Either**, you get \$100 for the first day, \$200 for the second day, \$300 for the third day. Each day you are paid \$100 more than the day before.

   - **Or**, you get 1 cent for the first day, 2 cents for the second day, 4 cents for the third day. Each day you are paid double what you were paid the day before.

   Which one will you prefer?

3. Find the value of:

   Example: $3^3 = 81$ $(power\ of\ a\ number)$, $(-3)^3 = -81 (power\ of\ a\ number)$;
   $-3^3 = -81$ $(opposit\ to\ the\ power\ of\ a\ number)$

   2^5; $(-2)^5$; -2^5; 3^4; $(-3)^4$; -3^4
   0.2^6; $(-0.2)^6$; -0.2^6; 0.05^2; $(-0.05)^2$; -0.05^2

10^7; $(-10)^7$; -10^7; 0.1^3; $(-0.1)^3$; -0.1^3

Which of these expressions are 'powers of a number,' and which are 'numbers opposite to the power of a number'?

4. Express the following numbers as power:
128; -128; 0.0016; -0.0016; 0.0009; -0.0009

5. Compare the values of the following expressions
$a.\ 3^{15}\ and\ 4^{15}$; $b.\ (-3)^{15}\ and\ (-4)^{15}$ $c.\ 9^{24}\ and\ 7^{23}$ $d.\ (-9)^{24}\ and\ (-7)^{23}$
$e.\ 1.8^5\ and\ 1.8^6$; $f.\ (-1.8)^5\ and\ (-1.8)^6$ $c.\ \left(\frac{2}{3}\right)^7\ and\ \left(\frac{2}{3}\right)^8$ $d.\ \left(-\frac{2}{3}\right)^7\ and\ \left(-\frac{2}{3}\right)^8$

6. Write as a power:
Example:
$\frac{1}{2} = 2^{-1}$; $\frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

$\frac{1}{3}$; $\frac{1}{25}$; $\frac{1}{27}$; $\frac{1}{125}$;

7. Without doing calculations, prove that the following inequalities hold:
Example:
$39^2 < 2000$: $39 < 40$, $39^2 < 40^2 = 1600$; $1600 < 2000$.
$a.\ 29^2 < 1000$; $b.\ 48^2 < 3000$; $c.\ 42^2 > 1500$; $d.\ 67^2 > 3500$

8. Compare the following exponents:
$a.\ 2^{10}\ \ and\ \ 10^3$; $b.\ 10^{100}\ \ and\ 100^{10}$
$c.\ 2^{300}\ \ and\ \ 200$; $d.\ 31^{16}\ \ and\ 17^{20}$; $e.\ 4^{53}\ \ and\ 15^{45}$

9. A particular lake has water lilies growing on it. On the first day, there is one water lily. Each day, the number of water lilies doubles. After 30 days, the water lilies cover half the lake. How long before they also cover the other half of the lake, so the whole lake is full?

10. Write as a power:
$a.\ 2^3 + 2^3 + 2^3 + 2^3$; $b.\ 3^4 + 3^4 + 3^4$;

11. $x^5 < y^8 < y^3 < x^6$
Where 0 should be placed?