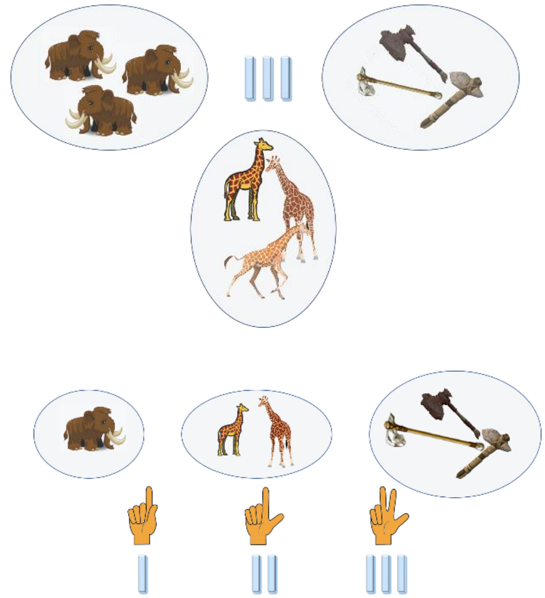
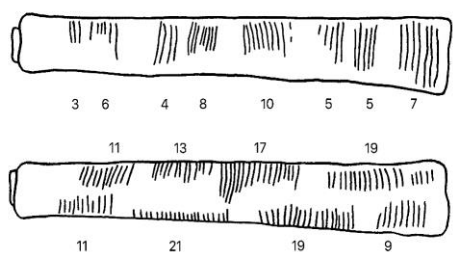
Chapter 1.



A very long time ago, our ancestors realized that each group of objects possesses a quantitative property: how many objects are there in the group? This property doesn't depend on the nature of the objects themselves. Groups can be compared based on this property, determining which group has more and which group has fewer items. This gave rise to the concept of "number" (now we use the term natural numbers for numbers to count objects). Initially, prehistoric people compared the number of objects in the group with the count of fingers– since hands were ever-present! Then they began scratching marks on wood and bones as a means to record quantities. One of the oldest known examples of such bones is the Ishango bone, which dates back to around 20,000-25,000 years ago.

Historians continue to debate the purpose of the Ishango bone. Some think that might have been an early calculating tool due to its tally marks grouped in a specific manner. (Of course, we cannot possibly know how exactly it was used.)



The next step in the progress of math is the creation of the arithmetic operations. What can we do with numbers? Numbers can be added together. At first, the operation +1 was developed:

||| + | = ||||

(There were no "+" sign, but we can use it for convenience.)
Then addition of two groups made the big progress:

||||||| + ||| = ||||||||||

All the numbers we now use to count are called natural numbers. In the later times, various systems of writing numbers were developed, and in our present decimal system we can write the same as:

$$3 + 5 = 8$$

$$\underset{addend}{3} + \underset{addend}{5} = \underset{sum}{8}$$

$$\underset{minuend}{8} - \underset{subtrahend}{3} = \underset{difference}{5}$$

$$\underset{minuend}{8} - \underset{subtrahend}{5} = \underset{difference}{3}$$

The operation of subtraction for natural numbers is the way to find the number that, when added to the numbers we are subtracting, results in the initial number. There are special names for the numbers in this operation:

$$addend + addend = sum$$

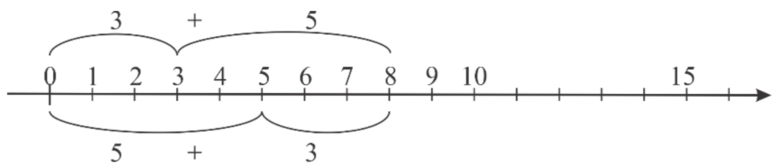
$$minuend - subtrahend = difference$$

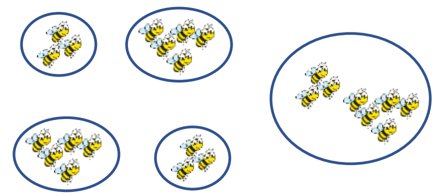
So, if we add subtrahend to a difference, the result should be the minuend.

The operation of addition has a few properties:

- it's commutative: it doesn't matter what addend goes first; the sum will not change.
- it's associative: if three terms are added together, it doesn't matter how they are added—whether the first two and then the third or the second plus the third and then the first—the result will remain the same.

Commutative and associative properties of addition are intuitively easy to understand.





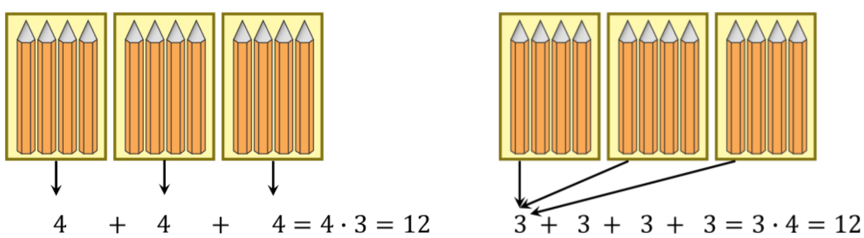
After that, multiplication was introduced, as addition of the same addend (term) several times.

$$4 \cdot 3 = \underbrace{4+4+4}_{3\ times} = \underbrace{3+3+3+3}_{4\ times} = 12$$

We can write it using variables, in a general form:

$$c \cdot b = \underbrace{c+c+\cdots+c}_{b\ times} = \underbrace{b+b+\cdots+b}_{c\ times}$$

The result of multiplication is called product, and the participants of the operation are called factors. c and b are factors, and a is a product.



How many pencils are there in three boxes, if there are four pencils in each box?

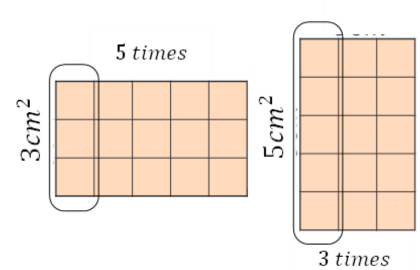
$3 \cdot 4 = 12$; three and four are factors, 12 is a product.

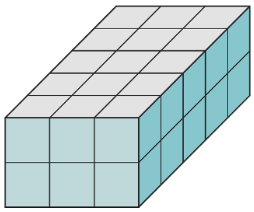
Multiplication also has properties:

- it's commutative. It doesn't matter what factor goes first; the product will not change.
- it's associative. If three factors are multiplied, it doesn't matter whether we first get the product of the first two and then multiply by the third factor or the second and third multiplied first, and then the result is multiplied by the first factor—the product will still be the same.

The commutative property of multiplication can be also illustrated by calculating of the area of a rectangle:

$$S = 3cm^2 \times 5times = 5cm^2 \times 3times = 15cm^2$$





The associative property can be shown by calculating of the volume of parallelepiped. How many cubed are stacked together into the parallelepiped?
First, we can multiply five by three to find out how many cubes are in the horizontal slice, and then multiply by 2 (the number of slices). Or multiplication of three by two will provide the number of cubes on the front slice, and there are five such slices.

$$(3 \cdot 5) \cdot 2 = (2 \cdot 5) \cdot 3 = (2 \cdot 3) \cdot 5 = 2 \cdot 3 \cdot 5 = 24$$

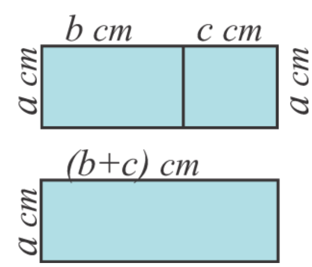
	Commutative	Associative	Identity
Addition	$a + b = b + a$	$(a + b) + c = a + (b + c)$	$a + 0 = a$
Multiplication	$a \cdot b = b \cdot a$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$	$a \cdot 1 = a;$ $a \cdot 0 = 0$
Distributive property of multiplication over addition	$a \cdot (b + c) = a \cdot b + a \cdot c$		

Distributive property can be illustrated with the following problem:
The farmer put green and red grapes into boxes. Each box contains 5lb of grapes. How many pounds of green and red grapes altogether did the farmer put into boxes if he had 10 boxes of green and 8 boxes of red grapes?
We can first find out how many boxes of grapes the farmer has altogether and multiply it by 5lb in each box, or we can find out the weight of white and red grapes in the boxes and then add it.

$$5 \times (10 + 8) \quad or \quad 5 \times 10 + 5 \times 8$$

$$5 \times 18 = 50 + 40; \qquad 90 = 90$$

Another example:



The combined area of these two rectangles is the sum of two areas, $S = a \times b + a \times c$
The rectangle with one side a cm and the other side $(b + c)$ cm will have exactly the same area, $S = a \times b + a \times c = a \times (b + c)$. (I have used the variables a, b, and c instead of numbers to show the distribution property in a general way).
Using the distributive property, we can factor out the common factor of two terms of the expression, for example:

$$6 \cdot 7 + 6 \cdot 3 = 6(7 + 3) = 6 \cdot 10 = 60$$

Properties of the arithmetic operations can be used to simplify the calculations.
Example:
We need to multiply a two-digit number by a single digit number.
$28 \cdot 9 = (20 + 8) \cdot 9 = 20 \cdot 9 + 8 \cdot 9 = 180 + 72$

$180 + 72 = 100 + 80 + 70 + 2 = 100 + 150 + 2 = 252$

Another example: how to find the value of the expression: $250 \cdot 61 - 25 \cdot 390$

Of cause, it can be calculated in a very straightforward way:

$250 \cdot 61 - 25 \cdot 390 = 15250 - 9750 = 5500$

Or we can use the properties of multiplication (which property has been used here?):

$25 \cdot 390 = 25 \cdot 39 \cdot 10 = 250 \cdot 39$

$250 \cdot 61 - 25 \cdot 390 = 250 \cdot 61 - 250 \cdot 39 = 250(61 - 39) = 250 \cdot 22 =$

$= 25 \cdot 22 \cdot 10 = (20 + 2) \cdot 25 \cdot 10 = (20 \cdot 25 + 2 \cdot 25) \cdot 10 = 550 \cdot 10 = 5500;$

Example of solving word problems:

On the first day, Mary read 21 pages. On the second day, she read twice as many pages as she did on the first day. On the third day, Mary read 15 pages less than she did on the second day. How many pages did she read altogether in three days?

This problem can be solved easily by steps:

$21 \cdot 2 = 42\ p.$ The number of pages Mary read on the second

$42 - 15 = 27\ p.$ The number of pages Mary read on the third day.

$21 + 42 + 27 = 90\ p.$ Total number of pages Mary read altogether.

Also, we can solve the problem by writing all the steps into the single numeral expression:

$$\underbrace{21}_{first\ day} + \underbrace{(21 \cdot 2)}_{second\ day} + \underbrace{(21 \cdot 2 - 15)}_{third\ day} = 21 + 42 + 27 = 90$$

Exercises:

1. Do the calculations in your head:

   $25 \cdot 8;$ $132 + 221;$ $248 - 134;$ $9 \cdot 38;$ $321 \cdot 41 + 59 \cdot 321;$

2. Using the equality, $678 + 1357 = 2035$ find the value of

   $a.\ 2035 - 1357;$ $b.\ 2035 - 678;$

3. Fill the empty boxes in the table (draw the table in your notebook):

a	16	28	44		49			
b	7	13		18		12	23	
$a + b$			60	67			833	72
$a - b$					35	60		0

4. In the number 3,728,954,106, remove three digits so that the remaining digits in the same order represent
   a. the smallest seven-digit number;
   b. the largest seven-digit number;

5. Find the unknown:

$a.\ b + 1111 = 3000;$ $b.\ 456 + c = 1362;$ $c.\ p + 222 = 152;$

$d.\ 1765 - n = 753;$ $e.\ a - 56 = 135;$ $f.\ l - 175 = 1889;$

$g.\ 2050 - f = 12;$ $h.\ 345 + m = 1214;1$

6. Write without parenthesis and evaluate, also evaluate with parenthesis:

Example: $123 - (12 + 5) = 123 - 12 - 5 = 106$

$123 - (12 + 5) = 123 - 17 = 106$

$a.\ 234 + (34 - 12);$ $b.\ 432 - (35 + 230);$ $c.\ 527 - (78 - 23).$

7. Evaluate (what is the best way to compute it? Hint: use commutative property):

Example: $3 \cdot 2 \cdot 7 \cdot 8 \cdot 5 = 2 \cdot 5 \cdot 3 \cdot 7 \cdot 8 = 10 \cdot 21 \cdot 8 = 10 \cdot 194 = 1680$

$a.\ 72 + 59 + 97 + 28 + 41;$ $b.\ 32 + 34 + 36 + 38;$

$c.\ 5 \cdot 19 \cdot 5 \cdot 3 \cdot 2 \cdot 2;$ $d.\ 715 - 99 - 101;$

$e.\ (629 + 56) - 629;$ $e.232 - (95 + 132);$

8. Evaluate by the most convenient way:

$a.\ 894 - (294 + 80);$ $b.\ (586 + 245) - 486;$ $c.\ (324 + 498) - 298$

9. Write without parenthesis, use the distributive property:

Example: $21 \cdot (2 + 3) = 21 \cdot 2 + 21 \cdot 3$

$a.32 \cdot (21 + 3);$ $b.\ 5 \cdot (21 - 3);$ $c.\ 7 \cdot (12 + 5);$ $d.\ 3 \cdot (45 + 15);$

10. Evaluate (what is the best way to compute it? Hint: use the distributive and/or commutative properties):

Example: $17 \cdot 55 + 45 \cdot 17 = 17 \cdot (55 + 45) = 17 \cdot 100 = 1700$

$a.\ 23 \cdot 15 + 15 \cdot 77;$ $b.\ 79 \cdot 21 - 69 \cdot 21;$

$c.\ 340 \cdot 7 + 16 \cdot 70;$ $d.\ 250 \cdot 61 - 25 \cdot 390;$

$e.\ 67 \cdot 58 + 33 \cdot 58;$ $f.\ 55 \cdot 682 - 45 \cdot 682$

$g.\ 5 \cdot 4 + 5 \cdot 44 + 5 \cdot 444 + 5 \cdot 4444;$

$h.\ 26 \cdot 25 - 25 \cdot 24 + 24 \cdot 23 - 23 \cdot 22 + 22 \cdot 21 - 21 \cdot 20 + 20 \cdot 19 - 19 \cdot 18 + 18 \cdot 17 - 17 \cdot 16 + 16 \cdot 15 - 15 \cdot 14;$

11. Using the commutative and associative properties of addition or multiplication simplify the expressions:

Example: $43 + b + 15 = 43 + 15 + b = 58 + b$

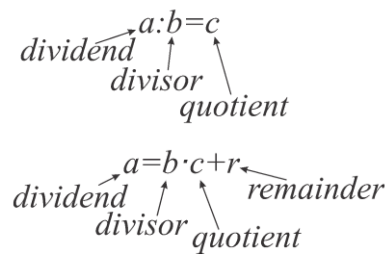
$a.\ 23 + a + 67;$ $b.\ 42 + b + 34 + 128;$

$c.\ 15 \cdot c \cdot 4;$ $d.\ 2 \cdot d \cdot 7 \cdot 5 \cdot 5 \cdot 2$

12. Compare the numbers with missing digits, if possible. If it's not possible, explain why:

$a.\ 9{*}{*} \ldots 2{*}{*}$ $b.\ 18{*}{*}{*} \ldots 20{*}{*}{*}$ $c.\ 3{*}{*}{*}4 \ldots 3{*}{*}{*}7$

$d.\ 6**** \ldots 6*5**$ $e.\ 9*4*4 \ldots 8*4*4$ $f.\ * *111 \ldots *1111$



13. Compare without doing actual calculations (put <, >, or =):
    - $a.$ $2453 + 235 \ \ldots \ 2453 + 236$
    - $b.$ $234 \cdot 123 \ \ldots \ 234 \cdot 122$
    - $c.$ $2341 - 123 \ \ldots \ 2341 - 122$
    - $d.$ $456 \div 4 \ \ldots \ 456 \div 3$
    - $e.$ $a \div 4 \ \ldots \ a \div 3$
    - $f.$ $b + 235 \ \ldots \ b + 236$

14. Find missing digits in the problems:

$$\begin{array}{r} \square 9\,5 \\ +\ 3\square 4 \\ \hline 8\,4\square \end{array} \qquad \begin{array}{r} \square 2\,0 \\ -\ 4\square 7 \\ \hline 3\,6\square \end{array}$$

15. Fill the empty spaces in the magic square3 so that all the columns, rows, and diagonals have the same sum of numbers.

a.

2		6
	5	1
4		

b.

3		15	14
13	16		
10	11		
8		12	9

16. A farmer prepared 12 liters of strawberry jam for the winter, raspberry jam — 4 liters less than strawberry, and apple jam — 2 times more than strawberry and raspberry together. How many liters of jam did the farmer prepare in total for the winter?

17. 1One book has 126 pages, and the other has 84 pages. Michle read both books in 5 hours. How much time did he spend reading each book if his reading rate did not change?

18. There are 160 notebooks in two boxes. In one box, there are 20 notebooks more than in the other. How many notebooks are in each box?

19. Alice was counting the steps of a staircase. Between the fifth and first floors, she counted 100 steps. How many steps are there between the first and second floors if the number of steps between all floors is the same?

20. Place parentheses into the following expression so that the statement is true.
    - $a.$ $15 - 35 + 5 \div 4 = 5$
    - $b.$ $60 + 40 - 16 \div 4 = 66$
    - $c.$ $24 \div 56 - 8 \times 4 = 12$
    - $d.$ $96 - 12 \times 6 \div 3 = 8$
    - $e.$ $64 \div 64 - 8 \times 4 = 2$
    - $f.$ $63 \div 9 + 54 = 1$
    - $g.$ $75 - 15 \div 5 + 10 = 22.$

21. In the expression place parentheses so that its value equals 10.

$$5 \cdot 8 + 12 \div 4 - 3$$

Chapter 2.

Multiplication and division.

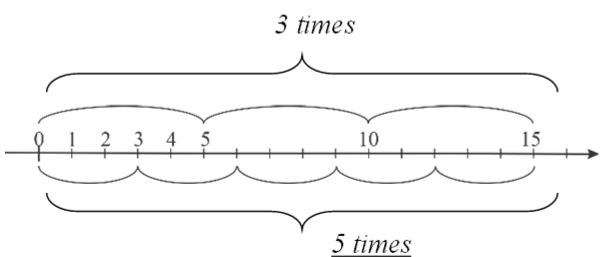
In part 1 we discussed a few properties of addition and multiplication. As we all know, multiplication is an arithmetic operation, equivalent to the repetitive addition of the same number.

$$c \cdot b = \underbrace{c + c + \cdots + c}_{b\ times} = \underbrace{b + b + \cdots + b}_{c\ times}$$

The result of multiplication is called the product, and the participants in the operation are called factors. c and b are factors, and a is a product.
Multiplication is closely connected with division; when we perform division of a number (this number is called the dividend) by a divisor, we are seeking a number (a quotient) that, when multiplied by the divisor, gives us the dividend.

(In this part of our course, we are discussing natural numbers, which are used for counting and start from 1: 1, 2, 3, and so on. I will omit the word 'natural' and use only the term 'number'.)
If there is a number c, that $c \times b = a$, then we can say that $a \div b = c$. This means that a is divisible by b, and b can be "fit" into a a whole number of times. c is also a factor of a, $a \div c = b$. For example,

$$3 \times 5 = 15; \quad 15 \div 3 = 5, \quad 15 \div 5 = 3$$



factor factor product

$$5 \cdot 3 = 5 + 5 + 5 = 3 + 3 + 3 + 3 + 3 = 15$$

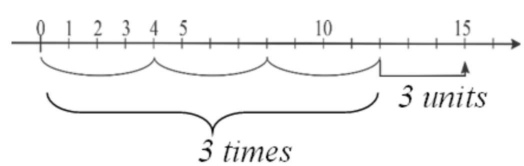
5 can fit into 15 exactly 3 times, 3 can go into 15 exactly 5 times.
15 is divisible by 3 and by 5.

If there is no number such that the divisor enters the dividend several times, then we can say that this number is not divisible by the divisor. In such cases, we can use division with a remainder.
For example, consider $15 \div 4$. 4 can't fully complete 15. It can fit into 12 three times, but there will be a little more left over. So,

$$15 \div 4 = 3\ with\ a\ remainder\ of\ 3$$

$15: 4 = 3R(3)$, or

$$15 = 4 \times 3 + 3$$



For division of any natural number by another, we can now write:

$$a \div b = cR(r), \qquad or \qquad a = b \times c + r$$

If $r = 0$, number a is divisible by number b.

Why can't we divide by 0? By definition, multiplying 0 by anything results in 0. Dividing by 0 would imply that there is a number that, when multiplied by 0, does not yield 0. But this is impossible. So, Therefore, division by 0 is undefined; it simply does not exist, and we cannot perform such an operation!

Example:
There are 48 kg. of grapes in 6 boxes. How many kilograms of grapes are there in 10 such boxes?
$48:6 = 8\ kg.$ How many kilograms are in one box.
$8 \cdot 10 = 80\ kg.$ How many kilograms are in 10 boxes.
Both steps can be combined to a single expression:
$48:6 \cdot 10 = 80\ kg.$

Divisibility rules.
Can we predict whether a given number is divisible by 2, 3, 4, and so on? There are divisibility rules:
Any (natural) number is divisible by 1.

- **A number is divisible by 2 if and only if its last digit is even or 0.**
- **A number is divisible by 3 if and only if sum of its digits is divisible by 3.**
- **A number is divisible by 4 if and only if the number formed by the last 2 digits is divisible by 4.**
- **A number is divisible by 5 if and only if its last digit is 5 or 0.**
- **What can you say about the divisibility rule for division by 6? Write it here:**
- **A number is divisible by 7 if and only if the result of subtracting twice the last digit from the remaining part of the number is also divisible by 7.**
- **A number is divisible by 8 if and only if the number formed by the last 3 digits is divisible by 8.**
- **A number is divisible by 9 if and only if the sum of its digits is divisible by 9.**
- **What can you say about the divisibility rule for division by 10?**
- **Number is divisible by 11 if and only if the result of alternation addition and subtraction is divisible by 11.**

Example:
Is number 517 divisible by 11?
$5 - 1 + 7 = 11$. 11 is divisible by 11, so 517 is also divisible.

Exercises.

1. Factorize (represent as a product of two or more factors, do not use 1 as a factor). Write one or more possible solutions:

Example: $35 = 3 \cdot 7$; $100 = 4 \cdot 5 \cdot 5$;

$$21, \quad 24, \quad 30, \quad 49, \quad 75, \quad 1000$$

2. Factorize (represent as a product of two or more factors, do not use 1 as a factor):

$$36; \quad 100; \quad 125; \quad 178; \quad 200.$$

3. If we want to divide a number by 7, what numbers can we get as a remainder?

4. Do the division, write your answer in a form $a: b = cR(r)$.

Examples:

$$25: 4 = 6R(1); \qquad 28: 7 = 4R(0)$$

$$a.\ 36: 5; \quad b.\ 43: 4; \quad c.\ 75: 3; \quad d.\ 126: 5; \quad 81: 9;$$

5. Evaluate the products and name the factors:
   Example: $3 \cdot 25 = 75, factors\ are\ 3\ and\ 25.$

$$a.\ 4 \cdot 12; \quad b.\ 7 \cdot 11; \quad c.\ 15 \cdot 20; \quad d.\ 34 \cdot 7;$$

6. The remainder of $1932 \div 17$ is 11, the remainder of $261 \div 17$ is 6. Is

$$2193 = 1932 + 261$$

   divisible by 17? Is it possible to say without division?

7. Find all natural numbers such that when divided by 5, the quotient and remainder are equal?

8. Factor out the common factor, find the value of the expressions:
   Example:

$$21 + 49 = 3 \times 7 + 7 \times 7 = 7 \times (3 + 7) = 7 \times 10 = 70$$

$$a.\ 35 - 25; \quad b.\ 44 + 77; \quad c.\ 81 - 45;$$

9. Will the sum and the product be even or odd for:
   a. 2 odd numbers
   b. 2 even numbers
   c. 1 even and 1 odd number
   d. 1 odd and 1 even number
   e. Explain why.

10. Fill the empty boxes in the table (draw the table in your notebook):

a	9	24	24		77		0	75
b	5	6		12		72	25	1
$a \cdot b$			168	96				
$a: b$					7	9		

11. Using the first equality, find the values of another two:

$a.\;\; 945{:}\,35 = 27; \qquad 27 \cdot 35 = ?; \qquad 945{:}\,27 = ?;$

$b.\;\; 555{:}\,15 = 37; \qquad 555{:}\,37 = ?; \qquad 15 \cdot 37 = ?;$

12. Find the unknown number:

$a.\;\; 18 \cdot x = 450; \qquad b.\;\; 1190{:}\,c = 34 \qquad c.\;\; 25 \cdot x = 1000$

$d.\;\; d \cdot 23 = 2346; \qquad e.\;\; n{:}\,17 = 22 \qquad f.\;\; 37 \cdot x = 851$

13. Find missing digits in the problems:

$$\begin{array}{r} 40\square 500 \\ \times \quad 8\square \\ \hline \square\square 28\square 0\square\square \end{array} \qquad\qquad \begin{array}{r} 2\square 7 \\ \times\; 6\square 2\square \\ \hline 43\square \\ \square\square 1 \\ +\;\square\square\square\square \\ \hline \square\square\square\square\square\square 0 \end{array}$$

14. What digit does the product end with:

a. the product of all single-digit numbers, excluding zero;
b. the product of all three-digit numbers;
c. the product of all hundred-digit numbers?

15. Calculate by grouping the identical terms:

9 + 5 + 5 + 9 + 9 + 5 + 9 + 5 + 9 + 5 + 9 + 5 + 5;
6 + 3 + 6 + 2 + 3 + 2 + 6 + 2 + 3 + 3 + 2 + 2.

16. Evaluate:

$a. 3 \cdot 5 \cdot 2 \cdot 7; \qquad b.\;\; 5 \cdot 5 \cdot 6 \cdot 4; \qquad c.\;\; 7 \cdot 2 \cdot 5 \cdot 2 \cdot 5$

$d. 2 \cdot 9 \cdot 5 \cdot 5 \cdot 4; \qquad e.\;\; 8 \cdot 4 \cdot 125 \cdot \qquad c.\;\; 5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 6;$

17. It is known, that $x \cdot y = 12$ What is the value of:

$a.\;\; x \cdot (y \cdot 5);$ b. $(x \cdot 2) \cdot y;$ $c.\;\; y \cdot (x \cdot 10);$ $d.\;\; (y \cdot 2) \cdot (x \cdot 3);$

18. Will the following numbers be divisible by

a. by 2:
   123457, 1029384756, 435672198745631578 30
b. by 3
   1347, 45632, 5637984265
c. by 5:
   5635, 78530, 657932, 45879515
d. by 7:
   1645, 234, 5478, 889, 16506

19. Write all divisors of numbers: 8, 12, 15, 36

Example: $D(8)\; are\; 1, 2, 4, 8$

20. Is the product of 1247 and 999 divisible by 3 (no calculations)?

21. Number a is divisible by 5. Is the product $a \cdot b$ divisible by 5?

22. Without calculating, establish whether the product is divisible by a number?

$a.\ \ 508 \cdot 12\ by\ 3$ $\qquad b.\ 85 \cdot 3719\ by\ 5$

$c.\ \ 2510 \cdot 74\ by\ 37$ $\qquad d.\ 45 \cdot 26 \cdot 36\ by\ 15$

$e.\ \ 210 \cdot 29\ by\ 3, by\ 29$ $\qquad f.\ 3800 \cdot 44 \cdot 18\ by\ 11, 100, 9$

23. Without calculating, establish whether the sum is divisible by a number:

$a.\ 25 + 35 + 15 + 45 \quad by\ 5;$ $\qquad b.\ \ 14 + 21 + 63 + 24\ by\ 7$

$c.\ 18 + 36 + 55 + 90 \quad by\ 9;$

24. How many vans are needed to take 55 students on a field trip if a van can take 12 students?

25. The summer vacation is 73 days long. Which day of the week will be the last day of vacations if the first day was Tuesday?

26. Show that among any three consecutive natural numbers, there will be one divisible by 3.

27. Among four consecutive natural numbers will be a number
   a. Divisible by 2?
   b. Divisible by 3?
   c. Divisible by 4?
   d. Divisible by 5?

28. The cinema has two auditoriums: a large one and a small one. The large auditorium has 40 rows, with 45 seats in each row. The small auditorium has 25 rows, with 24 seats in each row. How many times does the number of seats in the large auditorium exceed the number of seats in the small auditorium?

29. A bag containing 4 apples and 10 plums weighs 600 g, and a bag containing 2 apples and 10 plums weighs 400 g. How much does an apple weigh and how much does a plum weigh?

30. Compare without doing actual calculations:

$a.\ (30 + 56) \cdot 5\ and\ 30 \cdot 5 + 56 \cdot 5$ $\qquad b.\ \ (19 + 4) \cdot 7\ and\ 19 \cdot 7 + 10 \cdot 7$

$c.\ (14 - 7) \cdot 6\ and\ 16 \cdot 6 - 7 \cdot 6$ $\qquad d.\ \ (18 - 9) \cdot 7\ and\ 18 \cdot 7 - 11 \cdot 7$

$e.\ 6 \cdot 18 + 6 \cdot 21\ and\ (18 + 17) \cdot 6$ $\qquad f.\ \ 23 \cdot 15 - 5 \cdot 15\ and\ (23 - 7) \cdot 15$

Chapter 3.

Prime factorization.

In mathematics, factorization is a decomposition of a number or mathematical expression as a product of numbers or/and expressions.

A number can be represented as the product of two or more other numbers, for example:

$$40 = 4 \cdot 10 = 4 \cdot 2 \cdot 5, \quad 36 = 6 \cdot 6 = 2 \cdot 3 \cdot 6$$

A numerical expression can be written as a product:

$$7 \cdot 5 + 7 \cdot 3 = 7 \cdot (5 + 3)$$

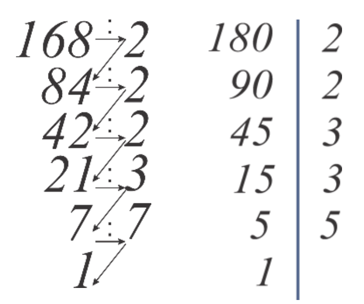
Is it possible for any natural number to be expressed as a product of 2 or more numbers other than 1 and itself?

Natural numbers, greater than 1 that has no divisors other than 1 and itself are called **prime numbers**.

Even numbers are the numbers divisible by 2 (they have 2 as a divisor), so they can be factorized as 2 times something else. Can an even number be a prime number? Is there any even prime number?

Prime factorization (or integer **factorization)** is a decomposition of a natural number into the product of prime numbers.

Any natural number has single unique prime factorization.



Prime factorization process:

Prime factors of 168 are $2, 2, 2, 3, 7$, and prime factors of 180 are $2, 2, 3, 3, 5$,

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 168; \quad 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 180$$

For Halloween, the Jonson family bought 168 mini chocolate bars and 180 gummy worms. What is the largest number of kids among whom the Jonsons can evenly divide both kinds of candy?

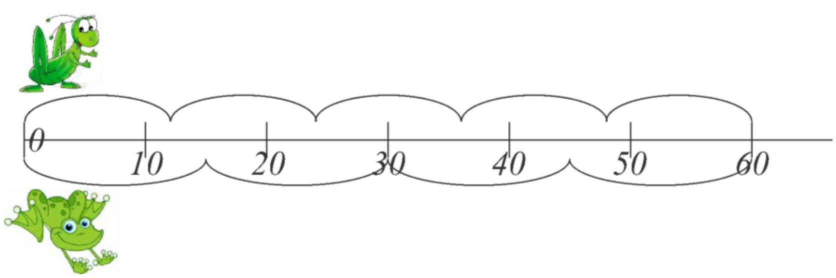
To solve this problem, we have to find a number that can serve as a divisor for 168 as well as for 180. There are several such numbers. The first one is 2. Both piles of candy can be evenly divided between just 2 kids. 3 is also a divisor. The Jonson family wants to treat as many kids as possible with equal numbers of candy. To do this, they have to find the Greatest Common Divisor (GCD), which is the largest number that can be a divisor for both 168 and 180.

Let's take a look at the set of all prime factors of 168 and 180. For 168 this set contains 2, 2, 2, 3, and 7. Any of these numbers, as well as any of their products, can be a divisor for 168. The same is true for the set of prime factors of 180, which are 2, 2, 3, 3, and 5. It is easy to see that these two representations have common

factors, 2, 2, and 3. This means that both numbers are divisible by any of these common factors and by any of their products. The largest product is the product of all common factors. This largest product has a name: Greatest Common Factor, or GCF. This GCF will also be the Greatest Common Divisor (GCD).
$GCF(168, 180) = 12$

$$\begin{array}{rcl} 2\cdot 2\cdot 2\cdot 3 & \cdot 7 & = 168 \\ 2\cdot 2\cdot 3\cdot 3\cdot 5 & & = 180 \end{array} \qquad \begin{array}{l} 168 \div 12 = 14 \\ 180 \div 12 = 15 \end{array}$$

Between 12 kids they can divide both kinds of candy evenly.
A grasshopper jumps a distance of 12 centimeters each leap, while a little frog covers a distance of 15 centimeters per leap. They both start at 0 and hop along the long ruler. What is the closest point on the ruler at which they can meet?
There are specific points on the ruler that both of them can reach after a certain number of leaps. One of these points is, of course, 180 cm.
A grasshopper would make 15 jumps, while a frog would make only 12. Will it be the only place where they can meet or are there other possible meeting points?
Any multiple of 180 will also be divisible by 12 and by 15. Are there any other common multiples of 12 and 15, which are less than 12×15 and are still divisible by 12 and 15?



$$\begin{array}{r|l} 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

Prime factorization of 12 and 15:

$$12 \cdot 15 = (2 \cdot 2 \cdot 3) \cdot (3 \cdot 5)$$

The number which we are looking for has to be a product of prime factors of 12 and 15.

$$\begin{array}{rl} 2\cdot 2\cdot 3 & = 12 \\ 3\cdot 5 & = 15 \end{array} \qquad 2\cdot 2\cdot 3\cdot 5 = 60 \qquad \begin{array}{rl} 7\cdot 2\cdot 2\cdot 2\cdot 3 & = 168 \\ 2\cdot 2\cdot 3\cdot 3\cdot 5 & = 180 \\ 7\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5 & = 2520 \end{array}$$

60 is the smallest number, which is divisible by both, 12 and by15, Least Common Multiple (LCM).

$$LCM(12, 15) = 60$$

The Johnson family wants to buy the same number of gammy worms and mini chocolates (168 mini per box and 180 gammy worms per box). How many boxes of each type of candy do they need to buy?
If all factors from one number are multiplied by the factors from the second number, which are missing in the first one, the resulting number is divisible by both numbers (is a common multiple) and is the smallest common multiple.
They need to buy $2520 \div 168 = 15$ boxes of mini chocolates and
$2520 \div 180 = 14$ boxes of gammy worms. Least common multiple for 168 and180 is 2520, which is much smaller than $168 \cdot 180 = 30240$.
Eratosthenes proposed a simple algorithm for finding prime numbers. This algorithm is known in mathematics as the Sieve of Eratosthenes, a simple, ancient algorithm for finding all prime numbers up to any given limit. It does so by iteratively marking as composite, i.e., not prime, the multiples of each prime, starting with the multiples of 2.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Exercises.

1. Prove that the following numbers are not prime:

   $a.\ 25;\qquad b.\ \ 8192;\qquad c. 99.$

2. Do the prime decomposition of the numbers: 66, 28, 128, 555, 1233

3. Find GCF:

   $a.\ GCF(8,48);\qquad b.\ GCF(7,15);\qquad c. GCF(20,1000);$

   $d.\ GCF(23,69);\qquad e.\ GCF(380,381);\qquad f. GCF(14,25);$

4. Find GCF using prime decomposition:

   $a.\ GCF(75,135);\qquad b.\ GCF(180,210);\qquad c. GCF(125,462);$

   $d.\ GCF(504,270);\qquad e.\ GCF(117,195,312);\qquad f. GCF(306,340,850);$

5. A teacher divided 87 notebooks between the students in the class equally. How many students in the class and how many notebooks did each student get?

6. Mary wrote down a sequence of multiples of a certain number, starting with the smallest one. The twelfth number in this sequence is 60. Find the first, sixth, and twentieth numbers?

7. How many multiples of 9 among first 100 (natural) numbers?

8. Find the LCM using the prime decomposition:

$a.\ LCM(28, 35);$ $b.\ LCM(16, 56);$ $c. LCM(21, 100);$

$d.\ LCM(18, 62);$ $e.\ LCM(264, 300);$ $f. LCM(360, 1020);$

$g.\ LCM(72, 90, 96);$ $h.\ LCM(58, 87, 435);$

9. A florist has 36 roses, 90 lilies, and 60 daisies. What is the largest number of bouquets he can create from these flowers, evenly dividing each kind of flowers between them?

10. There are less than 100 apples in a box. They can be evenly divided between 2, 3, 4, 5, and 6 kids. How many apples are there in the box?

11. Knives are sold 10 to a package, forks are sold 12 to a package, and spoons are sold 15 to a package. If you want to have the same number of each item for a party, what is the least number of packages of each you need to buy?

12. Fill in the table. Find a pattern. What can you say about GCF, LCM and a product of two numbers

Numbers	Product	GCD(GCF)	LCM
4 and 6	24	2	12
6 and 9			
5 and 7			
35 and 45			
16 and 18			
735 and 845	$735 \cdot 845$		

Can you explain what you noticed?

13. Two buses leave from the same bus station, following two different routes. The first bus takes 48 minutes to complete the round-trip route, while the second one takes 1 hour and 12 minutes to complete the round-trip route. How much time will it take for the buses to meet at the bus station for the first time after they have departed for their routes at the same time?

14. 1In the depot, 3 trains were formed from identical cars. The first one has 418 seats, the second one has 456 seats, and the third one has 494 seats. How many cars are in each train if it is known that the total number of cars does not exceed 50?

15. Mary has a rectangular backyard with sides of 48 and 40 yards. She wants to create square flower beds, all of the equal size, and plant different kinds of flowers in each flower bed. What is the largest possible size of her square flower bed?

16. Solve the problems:
   a. 57 apples were put into boxes, with 6 apples in each box. How many apples are left over?
   When peaches were put into 36 boxes, with 12 peaches in each box, 7 peaches were left over. How many peaches were there altogether?
   b. There were 120 candies at a party. When each kid took 4 candies, 12 candies were left over. How many kids were at the party?

17. Number 882 is divisible by 147, the number 147 is divisible by 21. Is 28 a divisor of 672?

18. Is number a divisible by number b? if yes, find the quotient.

$a.\ a = 2 \cdot 2 \cdot 3 \cdot 7 \cdot 7, \qquad b = 2 \cdot 2 \cdot 11$

$b.\ a = 2 \cdot 3 \cdot 5 \cdot 13, \qquad b = 5 \cdot 13$

$c.\ a = 3 \cdot 5 \cdot 5 \cdot 11 \cdot 17, \qquad b = 3 \cdot 5 \cdot 17$

$d.\ a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 19 \cdot 23, \qquad b = 2 \cdot 2 \cdot 3 \cdot 5$

$e.\ a = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 11 \cdot 13, \qquad b = 405$

$f.\ a = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 29, \qquad b = 2002$

19. What digit should be placed instead of * so that the number would be divisible by 9:

$a.\ 318*; \qquad b.\ *56; \qquad c.\ 48*25; \qquad d.\ 8*1;$

20. What digit should be placed instead of * so that the number would be divisible by 3:

$a.\ 318*; \qquad b.\ *56; \qquad c.\ 48*25; \qquad d.\ 8*1;$

21. Vinny the Pooh claimed that he knows the number, the product of whose digits is 6552. The Rabbit says it’s not true. How does he know?

22. The Johnsons family traveled from Stony Brook to Chicago. They covered the distance between these cities of 846 miles in 3 days. On Friday and Saturday, they covered 620 miles, on Sunday 53 miles more than on Saturday. How many miles did they drive on each of those days?

23. The Johnson family bought tomatoes, cucumbers, and onions on the farm, 18 kg altogether. How many kilograms of each vegetable did they buy if cucumbers were four times as much as onions, and tomatoes were as much as cucumbers?

24. Find missing digits in the problems:

$$\begin{array}{r} 35\square78 \\ +\ 4\square596 \\ 678\square \\ \hline 894\square5 \end{array} \qquad \begin{array}{r} 5\square728 \\ +\ 7045 \\ 83\square50 \\ 821\square\square \\ \hline 227165 \end{array}$$

25. Rebecca wants to decorate the box for her friend Alice's birthday present with a ribbon, as shown in the picture. How long should the ribbon be if she wants to leave 90 cm for the ends and the bow?

26. Find the unknown:

$a.\ (x + 24) + 53 = 930;$ $\qquad b.\ \ 35 + (x - 17) = 85$

$c.\ (34 - x) + 23 = 27;$ $\qquad d.\ \ 64 - (15 + x) = 28$

27. Can you tell without doing the actual calculation if \$150 is enough to buy a jacket priced at \$69, a pair of shoes (\$48), and a shirt priced at \$27?

28. Replace the addition with multiplication and evaluate:

Example:

$$\underbrace{150 + 150 + \cdots + 150}_{20\ times} + \underbrace{20 + 20 + \cdots + 20}_{150\ times} = 150 \cdot 20 + 20 \cdot 150 = 2 \cdot 20 \cdot 150 = 6000$$

$$\underbrace{10 + 10 + \cdots + 10}_{101\ times} + \underbrace{10 + 10 + \cdots + 10}_{101\ times}$$

29. A sequence of four numbers was recorded, each of which is 3 times larger than the previous one. The last number is 486. Find the first number.

30. A sequence of four numbers was recorded, each of which is 6 times smaller than the previous one. The last number is 2. Find the first number.

31. Find the values of both expressions and compare:

$a.\ \ 5 \cdot (8 + 14)\ \ and\ 5 \cdot 8 + 14;$ $\qquad b.\ \ 5 \cdot (6 + 4) \cdot 25\ and\ 5 \cdot 6 + 4 \cdot 25$

$c.\ \ 12 + 60 : (6 : 2)\ \ and\ \ (12 + 60) : 6 : 2;$ $\qquad d.\ \ 5 \cdot (20 - 6) + 40\ and\ 5 \cdot 20 - (6 + 40)$

32. Fill the empty spaces in the magic squares so that all the columns, rows, and diagonals have the same sum of numbers (all numbers should be different).

8		4
		9
	7	

	1	
3		7
	9	

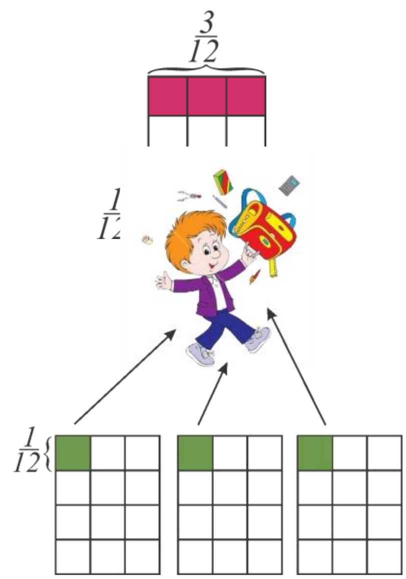
<u>Chapter 4.</u>

Fractions.

A fraction (from Latin: fractus, "broken") represents a part of a whole.

Look at the picture on the right:

the whole chocolate bar is divided into 12 equal pieces:

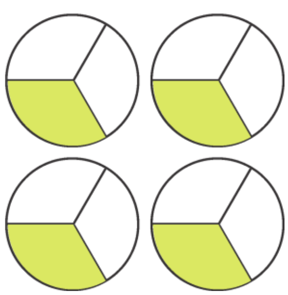


$$1\ (whole\ chocolate\ bar\ ): 12(equal\ parts) = \frac{1\ (whole\ chocolate\ bar\ )}{12(equal\ parts)}$$

$$= \frac{1}{12}(of\ the\ whole\ chocolate\ bar)$$

To divide 3 chocolate bars among 12 kids we can give each kid $\frac{1}{12}$ of each chocolate bar, altogether

$$\frac{1}{12} + \frac{1}{12} + \frac{1}{12} = 3 \times \frac{1}{12} = \frac{3}{12} = \frac{1}{4} = 3 \div 12$$



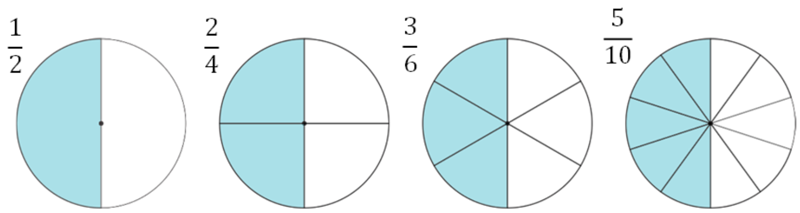
$$3 \div 12 = 3 \times \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$$

To divide 4 pizzas equally between 3 friends, we will give each friend $\frac{1}{3}$ of each pizza. Each friend will get $4 \div 3 = 4 \times \frac{1}{3} = \frac{4}{3}$, which is exactly 1 whole pizza ($3 \times \frac{1}{3} = \frac{3}{3} = 1$) and $\frac{1}{3}$.

When we talk about fractions, we usually mean the part of a unit. Proper fractions are parts of a unit; improper fractions are the sums of a natural number and a proper fraction. Sometimes we want to find a part of something which is not 1, but can be considered as a single object. For example, among 30 pencils, $\frac{2}{5}$ are yellow. How many yellow pencils are there? What does it mean to find $\frac{2}{5}$ out of 30? The whole pile of all these pencils is a single object, and we want to calculate how many pencils a little pile of $\frac{2}{5}$ of 30 contains. $\frac{2}{5}$ is 2 times $\frac{1}{5}$, and $\frac{1}{5}$ of 30 is $30 \div 5$. So $\frac{2}{5}$ of 30 pencils will be twice more:
$\frac{2}{5} \times 30 = 30 \div 5 \times 2$

Equivalent fractions.

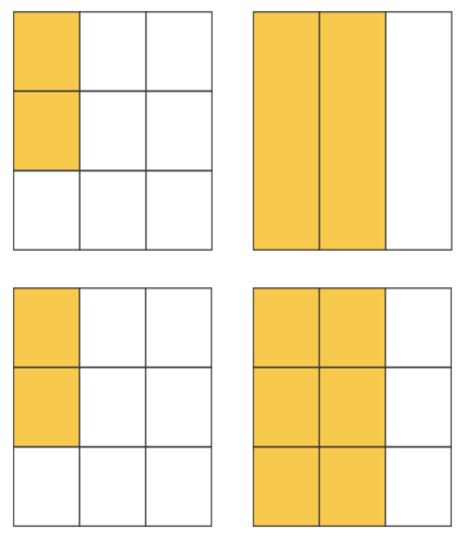
Some fractions can look different, but represent exactly the same part of the whole.



$$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{5}{10}; \qquad \frac{1}{2}=\frac{1\cdot 2}{2\cdot 2}=\frac{1\cdot 3}{2\cdot 3}=\frac{1\cdot 5}{2\cdot 5}$$

We can multiply the numerator and denominator of a fraction by the same number (not equal to 0), and the fraction will not change; it's still the same part of the whole. We are only dividing the whole into smaller parts and taking more such parts: if parts are twice smaller (denominator is multiplied by 2), we need twice more such parts to keep the fraction the same (numerator is multiplied by 2).
This property of fractions can be used to reduce fractions. If there are common factors in the numerator and denominator, both numbers can be divided by common factors.

$$\frac{25}{35}=\frac{5\cdot 5}{7\cdot 5}=\frac{5}{7}; \qquad \frac{77}{352}=\frac{7\cdot 11}{32\cdot 11}=\frac{7}{32}$$



Addition and subtraction of fractions with unlike denominators.
Let's try to add $\frac{2}{9}$ and $\frac{2}{3}$. What should we do? Why do we need to bring both fractions to the same denominator? We can add together only similar objects: apples to apples and oranges to oranges. Are two fractions $\frac{2}{9}$ and $\frac{2}{3}$ similar objects?

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3}, \qquad \frac{2}{9}=\frac{1}{9}+\frac{1}{9}$$

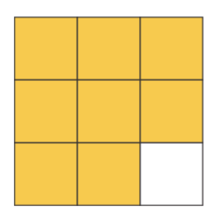
How we can add together

$$\frac{2}{9}+\frac{2}{3}=\frac{1}{9}+\frac{1}{9}+\frac{1}{3}+\frac{1}{3}$$

To be able to add two fractions we have to be sure that they have the same denominator. Each $\frac{1}{3}$ is exactly the same as $\frac{3}{9}$ and $\frac{2}{3}=\frac{6}{9}$

$$\frac{2}{3}=\frac{2\cdot 3}{3\cdot 3}=\frac{6}{9}$$

$$\frac{2}{9}+\frac{2}{3}=\frac{2}{9}+\frac{6}{9}=\frac{8}{9}$$



Mutually prime numbers are the numbers which do not have common factors, but 1. Like 8 and 9, they are both not prime, but do not have common factors other than 1.

If we multiply both numerator and denominator by the same number, the fraction will not change. Common denominator of both fractions should be the multiple of these denominators. If both numbers are prime (or mutually prime), the least common multiple is their product. If this is not the case, least common multiple is the simplest common denominator, but not the only one, any other multiple can do this task. Nominator and denominator of each fraction should be multiplied by a corresponding number to bring both fractions to a common denominator. For example,

$$\frac{3}{8}+\frac{5}{12}$$

Common denominator can be $8 \cdot 12 = 96$, but 24 is smaller.

$$\frac{3\cdot 3}{8\cdot 3}+\frac{5\cdot 2}{12\cdot 2}=\frac{9}{24}+\frac{10}{24}=\frac{19}{24}$$

How can fractions be compared? How can one know which fraction is grater and which is smaller? There are several ways to do it. First, fractions can be brought to a common denominator. For example, let's compare $\frac{7}{12}$ and $\frac{10}{18}$. The first fraction, $\frac{7}{9}$ is non-reducible. The second fraction can be reduced (it's a good idea to reduce fractions before doing anything):

$$\frac{10}{18}=\frac{2\cdot 5}{2\cdot 9}=\frac{5}{9}$$

Now, $\frac{7}{12}$ and $\frac{5}{9}$ can be brought to common denominator:

$$\frac{7}{12}=\frac{7\cdot 3}{12\cdot 3}=\frac{21}{36};\quad \frac{5}{9}=\frac{5\cdot 4}{9\cdot 4}=\frac{20}{36};\qquad \frac{21}{36}>\frac{20}{36}$$

The whole was divided into 36 equal parts and 21 such parts are greater than 20.
Another way to do it, is to bring them to a common numerator. Since 5 and 7 are both prime numbers, the LCM of them is their product:

$$\frac{7}{12}=\frac{7\cdot 5}{12\cdot 5}=\frac{35}{60};\quad \frac{5}{9}=\frac{5\cdot 7}{9\cdot 7}=\frac{35}{63};\qquad \frac{35}{60}>\frac{35}{63}$$

An equal number of parts are compared, but each part in the second case is smaller.
Also, both fractions can be compared with a third number, for example, $\frac{1}{2}$.

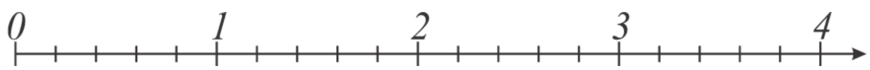
$$\frac{7}{12}=\frac{6}{12}+\frac{1}{12};\qquad \frac{10}{18}=\frac{9}{18}+\frac{1}{18}$$

Since, $\frac{7}{12}$ is greater than $\frac{1}{2}$ by $\frac{1}{12}$; and $\frac{10}{18}$ is greater than $\frac{1}{2}$ by $\frac{1}{18}$, $\frac{1}{12}>\frac{1}{18}$, so $\frac{7}{12}>\frac{10}{18}$.

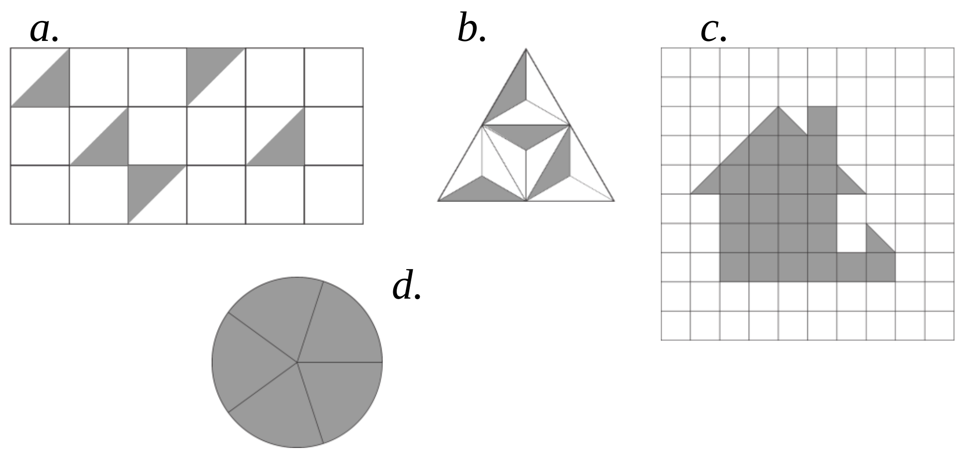
Exercises.

1. Mark the following fractions on the number line (draw the number line in your notebook):

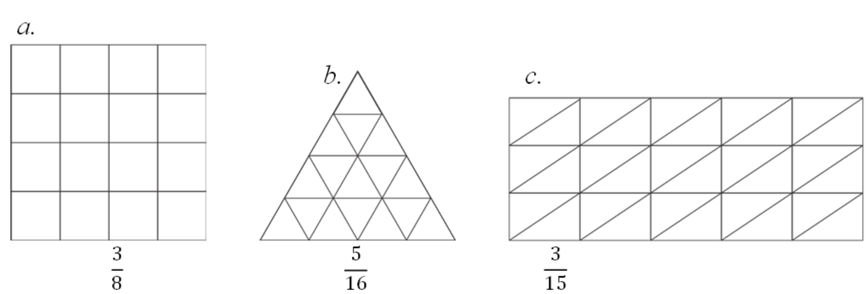
$$\frac{1}{5}, \quad \frac{3}{5}, \quad \frac{3}{3}, \quad \frac{7}{5}, \quad \frac{10}{5}$$



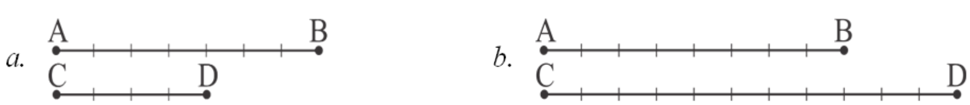
2. Write a fraction which show the shaded part of the shapes:



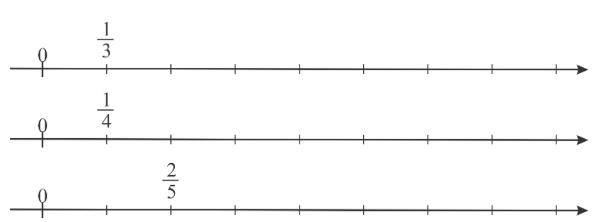
3. Shade the corresponding part of the figure (draw pictures in your notebook):



4. What part of the segment [AB] is the segment [CD]?



5. On the number lines, mark the number 1.



6. Draw a number line
   a. with a unit segment equal to 10 cells and mark the fractions:

$$\frac{1}{5};\ \frac{2}{5};\ \frac{3}{5};\ \frac{4}{5};\ \frac{5}{5};\ \frac{6}{5};\ \frac{7}{5};\ \frac{8}{5}.$$

b. with a unit segment equal to 12 cells and mark the fractions:

$$\frac{1}{4};\ \frac{1}{3};\ \frac{1}{2};\ \frac{2}{3};\ \frac{3}{4};\ \frac{2}{2};\ \frac{5}{4};\ \frac{5}{3}.$$

7. Fill the empty spaces for fractions:

$$\frac{2}{3}=\frac{\square}{9}=\frac{\square}{21}=\frac{4}{\square}=\frac{36}{\square}$$

8. Simplify (reduce) fractions:

Example: $\frac{7}{21}=\frac{1\cdot 7}{3\cdot 7}=\frac{1}{3}$

$$\frac{2}{8};\qquad \frac{14}{21};\qquad \frac{7}{49};\qquad \frac{3}{5};\qquad \frac{6}{8};$$

9. Bring the fractions to the common denominator:

$a.\ \frac{3}{5}\ and\ \frac{2}{3};\qquad b.\ \frac{3}{4}\ and\ \frac{5}{16};\qquad c.\ \frac{1}{4}\ and\ \frac{1}{6};$

10. Compare:

$a.\ \frac{3}{5}\ and\ \frac{4}{7};\qquad b.\ \frac{3}{5}\ and\ \frac{3}{8};\qquad c.\ \frac{3}{6}\ and\ \frac{1}{2};$

$d.\ \frac{1}{5}\ and\ \frac{5}{1};\qquad e.\ \frac{4}{12}\ and\ \frac{3}{4};\qquad f.\ \frac{2}{11}\ and\ \frac{1}{12};$

11. Evaluate:

$a.\ \frac{1}{5}+\frac{1}{2};\qquad b.\ \frac{2}{5}+\frac{3}{10};\qquad c.\ \frac{5}{9}-\frac{1}{3};$

12. What is bigger, the number c or $\frac{2}{3}$ of the number c? Why?

What is bigger, the number b or $\frac{3}{2}$ of the number b? Why?

What is bigger, $\frac{2}{3}$ of a number m or $\frac{3}{2}$ of a number m? Why?

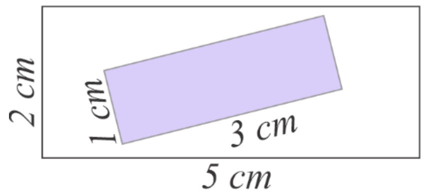
13.

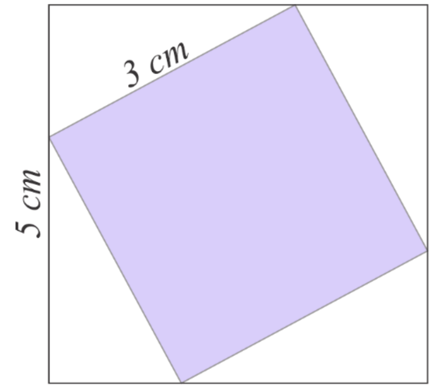
$a.$ $\frac{1}{7}$ of all students in the class is 4. How many students are there in the class?

$b.$ $\frac{2}{5}$ of all students in a class is 10. How many students are there in a class?

14. $\frac{5}{8}$ of a number is 15. What is the number?

15. What part of the
    a. big square the shaded square is?
    b. big rectangle the shaded rectangle is?





16. Write the answer as a fraction:
    a. Milk was evenly poured into 6 glasses. What fraction of the milk is in 1 glass? In 3 glasses? In 5 glasses?
    b. Candies were evenly distributed into 8 boxes. What fraction of the candies is in 1 box? In 3 boxes? In 7 boxes?

17. The pool fills with water in 5 hours. What fraction of the pool will be filled in 1 hour? In 2 hours? In 4 hours?

18. Write the answer as a fraction:
    a. In a bundle of 11 balloons: 3 of them are yellow, 4 are green, the rest are red. What fraction of all the balloons are red? Yellow? Green?
    b. In a box of 15 ballpoint pens: 7 of them are blue, 5 are black, the rest are green. What fraction of all the pens are blue and black together? Blue and green together?

19. The kilogram of cookies costs 15 dollars. How much Mary paid for $\frac{4}{5}$ of the kilogram of the cookies.

20. In the school cafeteria, there are 12 tables. There are 10 seats at each table. At the lunchtime $\frac{4}{5}$ of all sits were occupied by students. How many students were in the cafeteria?

21. An apple worm was eating an apple. On the first day it ate half of the apple, on the second day it ate half of the rest, and on the third day it ate half of the rest again. On the fourth day, it ate all the leftovers. What part of the apple did it eat on the fourth day?

22. Peter spent 2 hours doing his homework. $\frac{1}{3}$ of this time, he spent doing his math homework and $\frac{1}{4}$ of the remaining time he spent on the history assignment. How many minutes did Peter spent on his history assignment and how many minutes did he spent doing his math homework?

23. Write the expression for the following problems:
    a. 3 packages of cookies cost a dollars. How many dollars do 5 of the same packages cost?
    b. 5 bottles of juice cost b dollars. How many bottles can one buy with c dollars?

24. Half of the students of the class participated in a spelling bee competition. One third of them became winners. How many students are in the class, if there are 5 winners of the spelling bee in the class?

35. Find $a.\ \frac{3}{4}\ of\ 12,\quad b.\ \frac{2}{7}\ of\ 14,\quad c.\ \frac{5}{8}\ of\ 56$

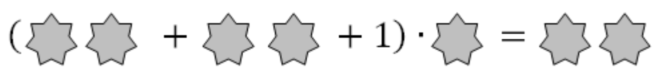
25. There are 48 pencils of each color: blue, yellow and green pencils, 72 red pencils and 120 coloring pictures. How many identical coloring sets can be created out of these pencils and pictures?

26. One book has 126 pages, and the other has 84 pages. Antony read both books in 5 hours. How much time did he spend reading each book if his reading speed did not change?

27. Are there four prime numbers such that the product of two of them is equal to the product of the other two?

28. There are bicycles and tricycles in a daycare center, 21 in total. The total number of wheels on all tricycles and bicycles is 55. How many tricycles and how many bicycles are there in the daycare? (Hint: how many wheels would be there if there were only tricycles?

29. Robert did his math assignment, but he stained his notebook. Each drop of ink covers the same digit, which is greater than 0. Please, restore his homework!



30. Find GCF (GCD) for numbers a and b:

$a.\ a = 2\cdot3\cdot3\cdot3\cdot5\cdot11,\qquad b = 2\cdot2\cdot3\cdot5\cdot7\cdot13$

$b.\ a = 2\cdot2\cdot5\cdot5\cdot7\cdot31,\qquad b = 2\cdot2\cdot3\cdot5\cdot5\cdot5\cdot31$

$c.\ a = 3\cdot7\cdot7\cdot19,\qquad b = 2\cdot5\cdot5\cdot11\cdot19;\qquad c.\ c = 2\cdot3\cdot3\cdot5\cdot7$

$d.\ a = 2\cdot2\cdot3\cdot3\cdot3\cdot29,\quad b = 3\cdot3\cdot5\cdot5\cdot29;\qquad c.\ c = 2\cdot3\cdot5\cdot11\cdot29$

Chapter 5.

Multiplication of a fraction by a number.

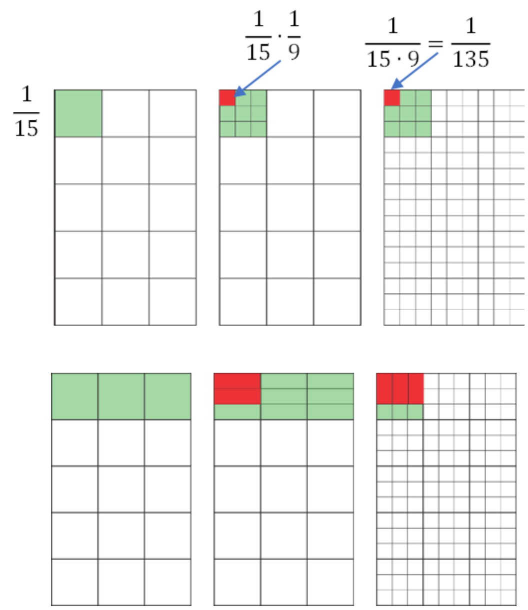
To multiply a fraction by a number, simply multiply the numerator by the number:

$$\frac{2}{7}\cdot 3=\frac{2}{7}+\frac{2}{7}+\frac{2}{7}=\frac{2+2+2}{7}=\frac{3\cdot 2}{7}=\frac{6}{7}$$

On the other hand:

$$\frac{2}{7}\cdot 3=3\cdot\frac{2}{7}=3:7\cdot 2=(3:7)\cdot 2=(3\cdot 2):7=\frac{3\cdot 2}{7}$$

Multiplication of a fraction by a fraction.
$\frac{1}{15}$ is a part of a whole divided into 15 equal small parts.



If we want to take $\frac{1}{9}$ part of this little $\frac{1}{15}$ chunk, we have to divide it into 9 even smaller pieces, to find $\frac{1}{9}$th of $\frac{1}{15}$th.

$$\frac{1}{15}:9=\frac{1}{15}\cdot\frac{1}{9}=\frac{1}{15\cdot 9}=\frac{1}{135}$$

If we need to take two small $\frac{1}{9}$ of $\frac{1}{15}$

$$\frac{1}{15}:9\cdot 2=\frac{1}{15}\cdot\frac{2}{9}=\frac{1\cdot 2}{15\cdot 9}=\frac{2}{135}$$

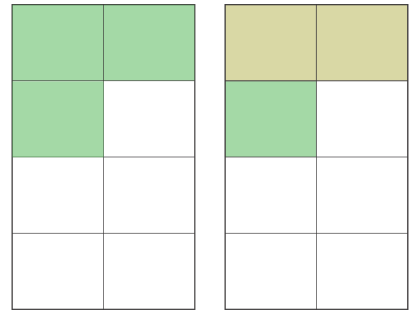
Or we want to find out $\frac{2}{9}$ of $\frac{3}{15}$.

$$\frac{3}{15}:9\cdot 2=\frac{3}{15}\cdot\frac{2}{9}=\frac{3\cdot 2}{15\cdot 9}=\frac{6}{135}$$

To multiply two fractions, we need to multiply numerators, multiply denominators and reduce fraction, if possible.
Examples:

$$\frac{3}{8}\cdot\frac{2}{7}=\frac{3\cdot 2}{4\cdot 2\cdot 7}=\frac{3\cdot 2}{4\cdot 7\cdot 2}=\frac{3}{4\cdot 7}=\frac{3}{28}$$

Division of fractions.



More of multiplication of fractions:

$$\frac{3}{8} \cdot \frac{2}{3} = \frac{2}{8} = \frac{1}{4}$$

By definition of the division operation, division of $\frac{1}{4}$ by $\frac{2}{3}$ should give the quotient $\frac{3}{8}$.

$$\frac{1}{4} : \frac{2}{3} = \frac{3}{8}$$

We can notice that the multiplication of $\frac{1}{4}$ by the inverse fraction $\frac{3}{2}$ will bring exactly $\frac{3}{8}$;

$$\frac{1}{4} : \frac{2}{3} = \frac{1}{4} \cdot \frac{3}{2} = \frac{3}{8}$$

To divide one fraction by another we need to multiply the dividend by the inverse fraction. Two fractions are inverse fractions if their product is 1. Inverse fractions can also be called reciprocal.

Two fractions are called *inverse fractions* if their product is equal to 1.

Examples:

$$\frac{1}{4} \cdot \frac{4}{1} = 1; \qquad \frac{3}{5} \cdot \frac{5}{3} = 1; \qquad \frac{4}{7} \cdot \frac{7}{4} = 1;$$

Exercise:

1. Evaluate:

$a.\ \frac{3}{8} + 2\frac{1}{4};$ $\quad b.\ 2\frac{1}{3} - 1\frac{1}{2};$ $\quad c.\ \frac{1}{4} + 3\frac{1}{6};$ $\quad d.\ 4\frac{1}{5} - 2\frac{3}{10};$

$e.\ 5\frac{5}{12} + 3\frac{2}{9};$ $\quad f.\ 7\frac{1}{9} - 4\frac{1}{3};$ $\quad g.\ 2\frac{4}{9} + \frac{1}{6};$ $\quad h.\ 2\frac{2}{7} - 1\frac{3}{5};$

$i.\ 4\frac{3}{5} + 10\frac{1}{4};$ $\quad j.\ 6\frac{1}{4} - 3\frac{2}{5};$

2. Evaluate:

Example:

$$4 - \frac{5}{6} = 3 + 1 - \frac{5}{6} = 3 + \frac{6}{6} - \frac{5}{6} = 3 + \frac{1}{6} = 3\frac{1}{6}$$

$a.\ 1 - \frac{1}{3};$ $\quad b.\ 1 - \frac{11}{20};$ $\quad c. 4 - \frac{1}{9};$ $\quad d.\ 6 - \frac{3}{7};$

$e.\ 1 - \frac{3}{4};$ $\quad f.\ 3 - \frac{1}{2};$ $\quad h. 5 - \frac{2}{5};$ $\quad i.\ 8 - \frac{2}{3};$

3. Evaluate:

$a.\ \ 5\frac{2}{6}-\frac{5}{6};$ $\quad b.\ \ 4\frac{5}{9}-\frac{8}{9};$ $\quad c.\ 6\frac{3}{7}-5\frac{5}{7};$ $\quad d.\ 2\frac{1}{3}-\frac{2}{3};$

$e.\ \ 3\frac{1}{12}-1\frac{5}{12};$ $\quad f.\ \ 4\frac{1}{8}-3\frac{5}{8};$

4. Write division operation as fraction, reduce fraction, if possible:

   Example:

$$4:8=\frac{4}{8}=\frac{1\cdot 4}{2\cdot 4}=\frac{1}{2}$$

$a.\ 3:7;$ $\quad b.\ 5:15;$ $\quad c.\ 3:9;$ $\quad d.\ 4:9;$ $\quad e.\ 12:13.$

5. Write fraction as a division, reduce fraction if possible:

   Example:

$$\frac{3}{12}=3:12=\frac{1}{4}=1:4$$

$a.\ \frac{4}{5};$ $\quad b.\ \frac{7}{9};$ $\quad c.\ \frac{5}{15};$ $\quad d.\ \frac{2}{8};$ $\quad e.\ \frac{11}{44}.$

6. Evaluate:

$a.\ \frac{4}{5}\cdot\frac{5}{7};$ $\quad b.\frac{2}{3}:\frac{5}{7};$ $\quad c.\frac{8}{9}\cdot\frac{3}{5};$ $\quad d.\ \frac{1}{4}:\frac{1}{2};$ $\quad e.\ \frac{9}{2}\cdot\frac{2}{9};$

$f.\frac{8}{21}\cdot\frac{7}{10};$ $\quad g.\frac{3}{4}:\frac{1}{2};$ $\quad h.\frac{8}{15}\cdot\frac{25}{28};$ $\quad i.\ \frac{5}{6}:\frac{7}{12};$ $\quad j.\ \frac{4}{9}:\frac{8}{9};$

7. Painter painted $\frac{2}{7}$ of the house in 4 days. How many days will it take him to paint the whole house?

8. Evaluate:

$a.\ \frac{3}{7}\cdot 2;$ $\quad b.3\cdot\frac{1}{6};$ $\quad c.\ 9\cdot\frac{5}{6};$ $\quad d.2\frac{1}{3}\cdot 2;$ $\quad e.4\cdot 1\frac{1}{2};$

9. Evaluate:

$a.\ \frac{1}{3}\cdot\frac{2}{7};$ $\quad b.\ \frac{1}{2}\cdot\frac{5}{6};$ $\quad c.\ \frac{1}{2}\cdot\frac{1}{3};$ $\quad d.\ \frac{3}{5}\cdot\frac{1}{2}\cdot\frac{4}{9};$ $\quad e.\ \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}$

$f.\ \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{23}{24}\cdot\frac{24}{25}$

10. Evaluate:

   Example:

$$3\frac{1}{4}\cdot 2=\left(3+\frac{1}{4}\right)\cdot 2=3\cdot 2+\frac{1}{4}\cdot 2=6+\frac{1\cdot 2}{4}=6+\frac{2}{4}=6+\frac{1}{2}=6\frac{1}{2}$$

Or it can be done by transferring the mixed number into the improper fraction:

$$3\frac{1}{4}\cdot 2=\frac{3\cdot 4+1}{4}\cdot 2=\frac{13}{4}\cdot 2=\frac{13\cdot 2}{4}=\frac{26}{4}=\frac{24+2}{4}=6+\frac{2}{4}=6+\frac{1}{2}=6\frac{1}{2}$$

$a.\ \ 2\frac{1}{3}\cdot 2;\qquad b.\ \ 4\cdot 1\frac{1}{2};\qquad c.\ \ 1\frac{1}{3}\cdot 9;\qquad d.\ \ \frac{3}{7}\cdot 2\frac{1}{3};\qquad e.\ \ 1\frac{1}{3}\cdot 1\frac{2}{2}$

$f.\ \ 12\cdot\frac{1}{6}\cdot 1\frac{1}{2}\cdot 3\frac{3}{4}\cdot 4\frac{1}{5};\qquad g.\ \ 3\cdot 5\frac{1}{4}\cdot 1\frac{1}{7}\cdot 5\frac{1}{2}\cdot\frac{4}{11};$

11. Write the fraction inverse to the given fraction:

Example:

$$\frac{3}{4}\rightarrow\frac{4}{3};\quad \frac{3}{4}\cdot\frac{4}{3}=1$$

$a.\ \frac{3}{7};\qquad b.\frac{7}{9};\qquad c.\ \frac{12}{5};\qquad d.\ \frac{9}{4};\qquad e.\frac{1}{3};\qquad f.\ \frac{1}{3};\qquad g.\ \frac{m}{n}\ (m,n\ \neq 0);$

12. Evaluate:

$a.\ \frac{2}{3}:\frac{5}{7};\qquad b.\ \frac{1}{4}:\frac{1}{2};\qquad c.\ \frac{3}{4}:\frac{1}{2};\qquad d.\ \frac{4}{9}:\frac{8}{9};\qquad e.\ \frac{5}{6}:\frac{7}{12};$

$f.\ 2:\frac{1}{7};\qquad g.\ 4:\frac{2}{3};\qquad h.\ 3:\frac{1}{2};\qquad i.\ 1:\frac{2}{7};\qquad j.\ 1:\frac{1}{4};$

13. Evaluate:

$a.\ \frac{4}{7}\cdot\frac{5}{24}:1\frac{1}{14};\qquad b.\ 25\cdot\frac{7}{15}:\frac{7}{9};\qquad c.\ \frac{7}{18}:\frac{20}{21}:\frac{5}{12};\qquad d.\ \frac{5}{9}\cdot 2\frac{1}{4}:20;$

14. Evaluate:

$a.\ 14:42;\qquad b.\ 2:3:5;\qquad c.\ 2:8\cdot 3;\qquad d.\ 100\cdot 6:40;\qquad e.\ 5:15\cdot 3.$

15. Melon weighs 7 pounds, and the watermelon is $1\frac{1}{5}$ times heavier. How many pounds is watermelon is heavier than the melon?

16. $4\frac{1}{2}$ kg. of candies were packed into $\frac{1}{2}$ kg packages. How many packages were the candies packed into?

17. Find the unknown:

$a.\ \frac{1}{3}\cdot x=\frac{1}{6};\qquad b.\ \frac{2}{5}\cdot x=1\frac{1}{5};\qquad c.\ \frac{2}{3}\cdot x=1$

$d.\ x\cdot 6=1\frac{1}{5};\qquad e.\ x\cdot 6=4;\qquad f.\ 3\cdot x=\frac{1}{3}$

18. There are 32 kg of apples in two baskets. There are 4 times as many apples in one basket as in the other. How many kilograms of apples are there in each basket?

19. While doing his homework, John noted the time spent on preparing each lesson: working with a map, solving a problem, and memorizing a poem. Using the data obtained, he formulated two problems. Solve them and try to create problems yourself using your own data:

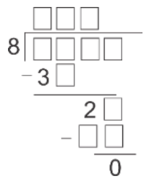
a. The student spent $\frac{1}{4}$ hour on geography and math problem, with the work on the map taking $\frac{1}{20}$ hour less than solving the problem. How much time was spent on each task?

b. The student spent $\frac{2}{5}$ hour on working with the map and memorizing the poem, with three times as much time spent on memorizing the poem as on working with the map. How much time did each task take?

20. The father is 40 years old. The son's age is $\frac{3}{8}$ of the father's age. How old is the son?

21. The son is 10 years old. His age is $\frac{2}{7}$ of the father's age. How old is the father?

22. Fill in the empty spaces:



23. Can you say which of the following statements are true and which are false?
    a. If a natural number is divisible by 4 and 3, it's divisible by 12
    b. If a natural number is divisible by 12, it's divisible by 3 and 4.
    c. If a natural number is divisible by 9, it's divisible by 3.
    d. If a natural number is divisible by 3, it's divisible by 9.

24. 25 identical thick books or 45 identical thin books can fit on a bookshelf. Will there be enough space on a bookshelf for 20 thick and 9 thin books?

25. Compare fractions:

$$a.\ \frac{1}{2}+\frac{1}{5}\quad and\quad \frac{1}{3}+\frac{1}{4};\qquad\qquad b.\frac{1}{2}-\frac{1}{3}\quad and\quad \frac{1}{4}-\frac{1}{5};$$

26. One natural number is 4 greater than the other. Find these numbers if their product is 96.

27. There are tents and cabins at the campground, with a total of 25 units. There are 4 people living in each cabin, and 2 people in each tent. How many tents and how many cabins are there at the campground if there are a total of 70 people staying there?

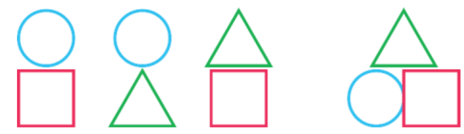
28. Find LCM and GCF of the numbers:

$$a. 72\ and\ 12;\qquad b. 16, 28, and\ 32;\qquad c. 8, 9\ and\ 25.$$

29. Poles are placed along the road, starting from point A, every 45 meters. It was decided to replace these poles with others, placing them 60 meters apart. Find the distance from point A to the nearest pole that will stand in the place of the old one, except for the pole at point A.

30. For a field trip, several buses were allocated to the schools, each with the same number of seats. 424 students from the elementary school went to the forest, and 477 students from the middle school went to the lake. All seats in the buses were occupied, and no one was left without a seat. How many buses were allocated, and how many passengers were on each bus?

31. Find the weight of each shape:



<u>Chapter 6.</u>

Complex fractions.

Complex fractions are formed by two fractional and/or numeral expressions, one on the top and the other one at the bottom, for example:

$$\frac{(2+3)\cdot 5}{7-\frac{1}{2}}; \qquad \frac{\frac{1}{2}+\frac{1}{3}}{\frac{7}{9}-\frac{2}{5}}$$

We know that the fraction bar is a just another way to write the division sign, so, the above expressions are equivalent to

$$\frac{(2+3)\cdot 5}{7-\frac{1}{2}} = \left((2+3)\cdot 5\right):\left(7-\frac{1}{2}\right); \qquad \frac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}+\frac{1}{4}} = \left(\frac{1}{2}+\frac{1}{3}\right):(\frac{2}{3}+\frac{1}{4})$$

It is easy to simplify a complex fraction:

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{2}{3}+\frac{1}{4}} = \left(\frac{1}{2}+\frac{1}{3}\right):\left(\frac{2}{3}+\frac{1}{4}\right) = \frac{\frac{3}{6}+\frac{2}{6}}{\frac{8}{12}+\frac{3}{12}} = \frac{\frac{5}{6}}{\frac{11}{12}} = \frac{5}{6}:\frac{11}{12} = \frac{5}{6}\cdot\frac{12}{11} = \frac{5}{1}\cdot\frac{2}{11} = \frac{10}{11}$$

Now let's simplify a little more sophisticated complex fraction:

Example 1.

$$\frac{1\frac{2}{3}+2\frac{4}{9}}{4\frac{26}{27}-2\frac{2}{9}} = \left(1\frac{2}{3}+2\frac{4}{9}\right):\left(4\frac{26}{27}-2\frac{2}{9}\right)$$

$$1\frac{2}{3}+2\frac{4}{9} = 1+\frac{2}{3}+2+\frac{4}{9} = 3+\frac{6}{9}+\frac{4}{9} = 3+\frac{10}{9} = \frac{27}{9}+\frac{10}{9} = \frac{37}{9}$$

$$4\frac{26}{27}-2\frac{2}{9} = 4+\frac{26}{27}-2-\frac{2}{9} = 2+\frac{26}{27}-\frac{6}{27} = 2+\frac{20}{27} = \frac{54}{27}+\frac{20}{27} = \frac{74}{27}$$

$$\frac{37}{9}:\frac{74}{27} = \frac{37}{9}\cdot\frac{27}{74} = \frac{37\cdot 3\cdot 9}{9\cdot 2\cdot 37} = \frac{3}{2} = 1\frac{1}{2}$$

Example 2:

$$\frac{3\frac{4}{7}:\left(6\frac{1}{28}-3\frac{3}{4}\right)}{\left(1\frac{5}{6}\cdot 1\frac{5}{22}\right):18\cdot 5}$$

First, let's find the value of the numerator:

$$3\frac{4}{7}:\left(6\frac{1}{28}-3\frac{3}{4}\right)=\frac{25}{7}:\left(5+1\frac{1}{28}-3-\frac{3}{4}\right)=\frac{25}{7}:\left(2+\frac{29}{28}-\frac{3}{4}\right)==\frac{25}{7}:\left(2+\frac{29}{28}-\frac{21}{28}\right)=\frac{25}{7}:\left(2\frac{8}{28}\right)$$
$$=\frac{25}{7}:\frac{64}{28}=\frac{25}{7}\cdot\frac{28}{64}=\frac{25\cdot 4}{64}=\frac{25}{16}$$

The value of the denominator is

$$\left(1\frac{5}{6}\cdot 1\frac{5}{22}\right):18\cdot 5=\left(\frac{11}{6}\cdot\frac{27}{22}\right)\cdot\frac{1}{18}\cdot 5=\frac{11\cdot 3\cdot 9}{2\cdot 3\cdot 11\cdot 2}\cdot\frac{1}{2\cdot 9}\cdot 5=\frac{5}{8}$$

Finally:

$$\frac{\frac{25}{16}}{\frac{5}{8}}=\frac{25}{16}:\frac{5}{8}=\frac{25}{16}\cdot\frac{8}{5}=\frac{5\cdot 5\cdot 8}{2\cdot 8\cdot 5}=\frac{5}{2}=2\frac{1}{2}$$

Problem solving examples:

What number x can be substituted with so that the fraction $\frac{x}{18}$ will be a nonreducible proper fraction?

For the fraction $\frac{x}{18}$ to be proper, x should be less than 18, and greater than 0. We have numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17. All of them greater than 0 and less than 18. For the fraction to be nonreducible, the numerator and denominator should not have common factors. 18 can be prime factorized as:

$$18=2\cdot 3\cdot 3$$

So, we have to exclude all even numbers, 2, 4, 6, 8, 10, 12, 14, 16.
The numbers left are 1, 3, 5, 7, 9, 11, 13, 15, 17.
Then we have to exclude all numbers divisible by 3: 3, 9, 15. Numbers that are not divisible by 3 will not be divisible by 6 $(2\cdot 3)$ and 9 $(3\cdot 3)$ as well. So, x can be substituted with 1, 5, 7, 11, 13, and 17. All these fractions are proper and nonreducible:

$$\frac{1}{18};\ \frac{5}{18};\ \frac{7}{18};\ \frac{11}{18};\ \frac{13}{18};\ \frac{17}{18};$$

56 tickets were sold for the flight, and 24 seats remained unoccupied. What fraction of the seats are occupied?

Step 1. Total number of seats is $56+24=80$
Step 2. One seat is $\frac{1}{80}$ part of the total number of seats. 56 such parts are $\frac{56}{80}$.

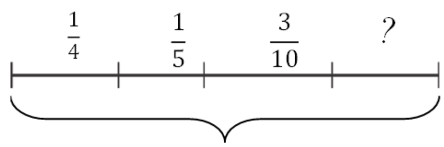
$$56:80=\frac{56}{80}=\frac{7\cdot 8}{10\cdot 8}=\frac{7}{10}$$

Answer: $\frac{7}{10}$ of all seats are occupied.

A cyclist covered $\frac{1}{4}$ of the distance in the first hour. During the second hour he drove $\frac{1}{5}$ of the distance, and during the third hour he covered $\frac{3}{10}$ of the distance. Which part of the planned distance does he still need to cover?

We need to add

$$\frac{1}{4}+\frac{1}{5}+\frac{3}{10};$$



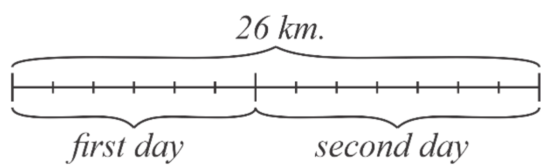
The common denominator is 10,

$$\frac{1}{4}+\frac{1}{5}+\frac{3}{10}=\frac{1\cdot 5}{4\cdot 5}+\frac{1\cdot 4}{4\cdot 5}+\frac{3\cdot 2}{10\cdot 2}=\frac{5+4+6}{20}=\frac{15}{20}=\frac{3}{4}$$

$$1-\frac{3}{4}=\frac{4}{4}-\frac{3}{4}=\frac{1}{4}$$

A backpacker walked 26 kilometers in 2 days. On the first day, he walked 6/7 of the distance he walked on the second day. How many kilometers did he walk each day?

On the first day, he walked exactly 1/7 less than on the second day. If we divide the distance he walked on the second day into 7 equal parts, we need to take 6 of those parts to determine how far he walked on the first day.



One seventh part of the second day's distance is

$$26:(6+7)=\frac{26}{6+7}=2\ km$$

On the first day a backpacker walked $2\cdot 6=12\ km$ and on the second day he walked $2\cdot 7=14\ km$.

Exercises:

1. Evaluate:

$$a.\ \frac{6}{1-\frac{1}{3}};\qquad b.\ \frac{1-\frac{1}{6}}{2+\frac{1}{6}};\qquad c.\ \frac{\frac{1}{2}+\frac{3}{4}}{\frac{1}{2}};\qquad d.\ \frac{\frac{7}{10}+\frac{1}{3}}{\frac{7}{10}+\frac{1}{2}};\qquad e.\ \frac{2-\frac{\frac{1}{2}-\frac{1}{4}}{2}}{2+\frac{\frac{1}{2}-\frac{1}{4}}{2}}$$

2. Write the expressions as fractions and evaluate:

$a.\ 14:42;\quad b.\ 2:3:5;\quad c.\ 2:8\cdot 3;\quad d.\ 100\cdot 6:40;\quad e.\ 5:15\cdot 3$

$f.\ (21\cdot 18):14;\quad g.\ 50:(16\cdot 25);\quad h.\ (12\cdot 15):40;\quad i.\ (4\cdot 24):(2\cdot 8)$

3. Write the expressions as fractions and evaluate:

$a.\ (3\cdot3\cdot5\cdot11):(3\cdot11);$ $\quad b.\ (2\cdot2\cdot3\cdot5\cdot7):(2\cdot3\cdot7);$

$c.\ (2\cdot3\cdot7\cdot13):(3\cdot7);$ $\quad d.\ (3\cdot5\cdot11\cdot17\cdot23(:(3\cdot11\cdot17);$

4. Evaluate:

$$\frac{3\frac{5}{11}\cdot6\frac{3}{4}}{3\frac{5}{11}\cdot6\frac{3}{4}+3\frac{5}{11}\cdot1\frac{1}{2}}$$

5. Into how many equal $\frac{1}{5}$ kilogram portions 5 kg cake can be divided?

6. The area of a rectangle is $\frac{5}{7}\ m^2$. One side of this rectangle is $\frac{3}{4}\ m$. What is the length of the other side?

7. Julia and Mary ate all the Halloween candy. Mary claimed that she ate $\frac{2}{3}$ of the candies, and Julia said that she ate $\frac{3}{5}$. Their parents think that something is wrong. Are they right?

8. The model of the house is $\frac{1}{25}$ of its real size. The width of a window on the model is 5 cm. How wide is a window in a real house?

9. What is the length of a segment if
   a. $\frac{2}{5}$ of its length is 12 meters;
   b. $\frac{3}{4}$ of its length is 9 centimeters;
   c. $\frac{3}{5}$ of its length is 15 millimeters.
   d. $\frac{2}{7}$ of its length is 8 meters.

10. What number x can be substituted with so that the fraction $\frac{x}{12}$ will be a nonreducible proper fraction?

11. From 42 m of fabric, 10 identical duvet covers were sewn, and from 33 m - 15 identical sheets. How much fabric is needed for a set that includes 1 sheet and 1 duvet cover?

12. Mary's 10 steps are 9 meters, while Julia's 20 steps are 17 meters. Whose steps are longer?

13. The sum of all numbers in each square is 10. What number should be placed instead of "?" ?

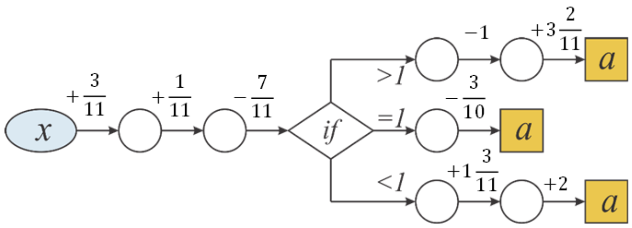
$2\frac{1}{7}$	$5\frac{4}{7}$
$\frac{3}{7}$	?

$1\frac{4}{5}$	$3\frac{2}{5}$
?	$2\frac{1}{5}$

$\frac{5}{9}$	?
$2\frac{7}{9}$	$1\frac{2}{9}$

?	$6\frac{8}{11}$
$\frac{2}{11}$	$2\frac{5}{11}$

14. Using the given algorithm, fill the table:



x	$\frac{6}{11}$	$\frac{14}{11}$	$\frac{4}{11}$	$\frac{3}{11}$	$1\frac{7}{11}$
a					

15. Find the unknown:

$$a.\ \frac{3}{5}+x=\frac{4}{5}; \qquad b.\ y-\frac{2}{7}=\frac{6}{7};$$

$$c.\ \frac{11}{25}-k=\frac{7}{25}; \qquad d.\ s+\frac{2}{9}=\frac{1}{9}+\frac{7}{9};$$

16. The weight of a turkey is three times less than the weight of a sheep, and the weight of three such sheep is 60 kg more than the weight of five turkeys. What is the weight of one turkey, and what is the weight of one sheep?

17. Solve the following problems:
   a. How many different 2-digit numbers (excluding numbers with repeating digits, like 44) can be created using the digits 1, 2, 3, and 4?

   b. How many teams of two students can be formed from 4 students: Mary, Elisabeth, John, and Mickle?

What is similar, and what is different in these problems?

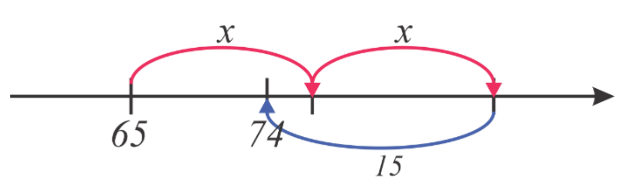
18. Fill in the empty spaces:

$$\begin{array}{r} 318 \\ \times\ \square\square \\ \hline \square\square 90 \\ +\ \square 54\ \\ \hline \square\square\square\square\square \end{array}$$

19. Without performing the multiplication, arrange the products in ascending order:

$$56\cdot 24;\quad 56\cdot 49;\quad 13\cdot 24;\quad 13\cdot 11;\quad 74\cdot 49;\quad 7\cdot 11.$$

20. Find x on the picture below:



21. Write five values for x such that:

$$\frac{1}{4} < x < \frac{1}{2}$$

22. Evaluate:

Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6};$$

One way to solve this is by bringing all fractions to a common denominator, 60:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 1 - \frac{30}{60} + \frac{20}{60} - \frac{15}{60} + \frac{12}{60} - \frac{10}{60} = 1 + \frac{32}{60} - \frac{55}{60} = \frac{60}{60} - \frac{55}{60} + \frac{32}{60}$$

$$= \frac{5}{60} + \frac{32}{60} = \frac{37}{60};$$

Another way to calculate it is to multiply the whole expression by the LCM of all denominators to bring everything to whole numbers:

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) \cdot 60 = 60 - 30 + 20 - 15 + 12 - 10 = 37$$

But the answer is now 60 times larger than the one we're looking for.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 37 : 60 = \frac{37}{60}$$

$a.\quad \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{18};$ $\qquad b.\quad \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$

23. If 1 is added to both the numerator and denominator of an irreducible fraction, will it necessarily remain irreducible? Can it remain irreducible?

24. If 1 is added to both the numerator and denominator of a reducible fraction, can it remain reducible? Can it become irreducible?

25. Do the calculations with time units:

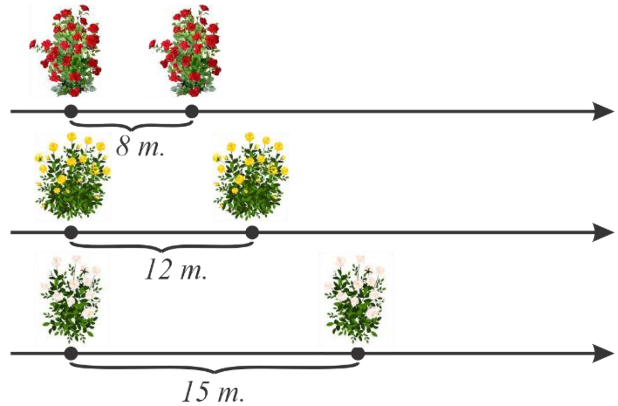
$a.\quad 18min.36s. + 24min.58s.;$ $\qquad b.\quad 5h.17min. - 3h.45min;$

$c.\quad 42min.48s. : 24;$ $\qquad d.\quad 3h.25min. \cdot 7;$

26. A giraffe is 12 times heavier than a kangaroo, and an elephant is 5 times heavier than the giraffe. What is the weight of each of these animals if their total weight is 5 tons 110 kg? ($1ton = 1000\ kg$.

27. *The inhabitants of the Unknown Planet, divide the day into several hours, an hour into several minutes, and a minute into several seconds. However, on their planet, there are 77 minutes in a day and 91 seconds in an hour. How many seconds are in a day on the Unknown Planet?

28. In a garden, there are red, yellow, and white rose bushes. At the beginning of the garden, red, yellow, and white bushes are planted in a row. Then, red roses are planted every 8 meters, yellow roses every 12 meters, and white roses every 15 meters, continuing until all three types of bushes are again planted in a row at the end of the garden. How long is this garden, and how many red, yellow, and white roses are planted there?



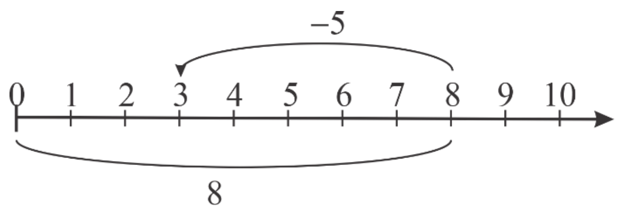
Chapter 7.

Positive and negative numbers.

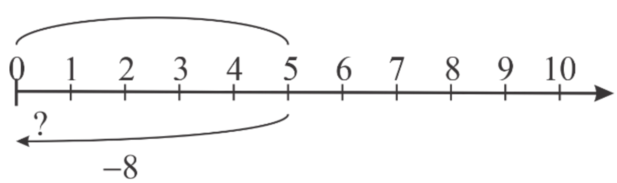
We have already observed that not every natural number can be divided by any other natural number. There are prime numbers, which can only be divided by 1 or by themselves. Other numbers are not prime, and are divisible by some numbers but not by others. To handle this we introduce a new kind of numbers – fractions – which are not natural numbers, but represent the parts of a whole.

What can we say about the operation of subtraction? Can we subtract any natural number from any other natural number? It's easy to subtract 5 from 8, but what happens if we need to subtract 8 from 5? Let's consider one example:

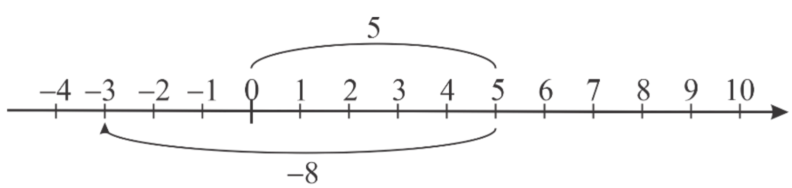
John has 8 dollars, and he wants to buy a 3-subject notebook, that costs 5 dollars. How much money will he have left?



Now, what if he only has 5 dollars, but he needs to buy a 5-subject notebook, priced 8 dollars?

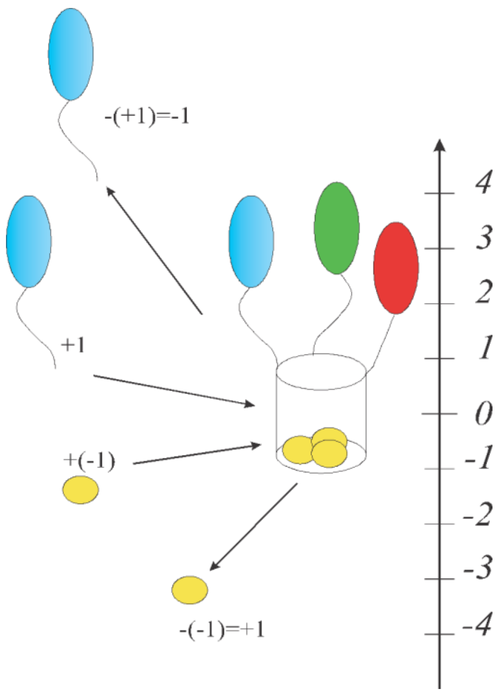


In this case, he can borrow 3 dollars from his friend, resulting in a balance of (-3) dollars (and a notebook).

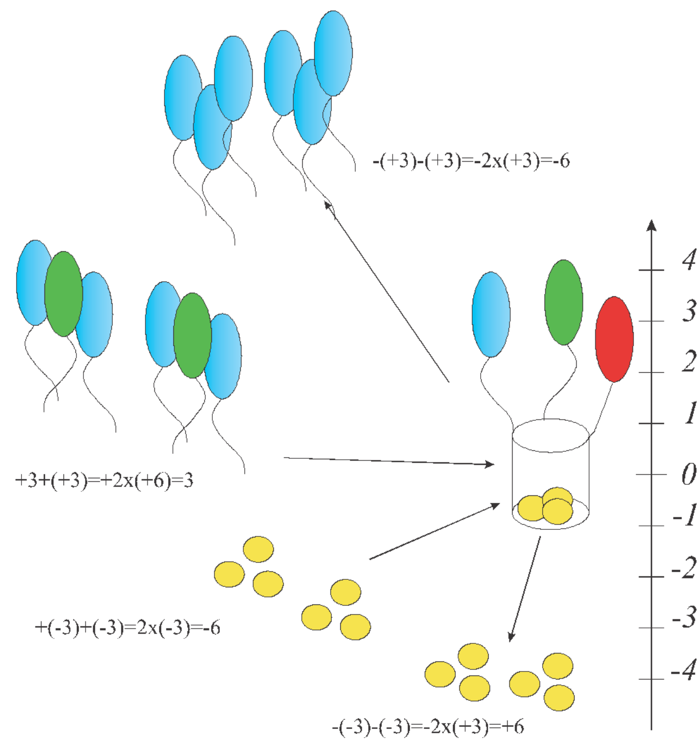


Another example of a negative numbers in our everyday life:

Temperature can be measured in different scales. In the USA we use Fahrenheit, in many other countries the Celsius scale is adopted. It based on a two reference points, water freezing temperature at 0°C and water boiling temperature at 100°C, 1°C is a 100th part of the 100°.



The Fahrenheit scale was developed based on three fixed points of temperature. Originally, Fahrenheit chose the freezing temperature of a mixture of ice, water, and ammonium chloride as 0°F, the freezing point of water as 32°F, and the average human body temperature as 96°F. The scale was later adjusted so that water boils at 212°F, and the average human body temperature was redefined as approximately 98.6°F.



Negative numbers represent opposites. If “positive” represents movement to the right, “negative” represents movement to the left. If “positive” represents above sea level, then “negative” represents below sea level. If “positive” represents a deposit, “negative” represents a withdrawal. They are often used to represent the magnitude of a loss or deficiency. Negative numbers appeared for the first time in history in the Nine Chapters on the Mathematical Art, which in its present form dates from the period of the Chinese Han Dynasty (200 BCE – 220 CE 200), but may well contain much older material. Liu Hui (c. 3rd century) established rules for adding and subtracting negative numbers. By the 7th century, Indian mathematicians such as Brahmagupta were describing the use of negative numbers. Islamic mathematicians further developed the rules of subtracting and multiplying negative numbers and solved problems with negative coefficients. Western mathematicians accepted the idea of negative numbers by the 17th century. Prior to the concept of negative numbers, mathematicians such as Diophantus considered negative solutions to problems "false" and equations requiring negative solutions were described as absurd.
We can use the very clear illustration of how positive and negative numbers work. At the beginning, the basket has a few balloons attached and the same number of sand bags (weights) placed inside, is positioned at 0 and doesn’t move. Balloons represent positive units; sand bags represent negative units. If we add one balloon, the basket gets pulled up by one unit. If we add one sand bag, the basket will be pulled down by one unit. If we remove one balloon, the basket will go one unit down, which is equivalent of adding one sand bag. So $-(+1) = +(-1)$. If we remove one sand bag, the basket will go one unit up, which is equivalent of adding one balloon. So $-(-1) = +(+1)$.
Two numbers that have the same magnitude but are opposite in signs are called opposite numbers.

Multiplication and division of negative numbers.

If we multiply 2 positive numbers, we will get a third positive number. What will happen if we multiply one negative and one positive number. Let's again review our model. In this case, we will add or remove our balloons and sand bags by groups of three. Addition of two groups of 3 sand bags will drive the basket 6 units down, because we add 6 bags. So $2 \times (-3) = -6$.

Removing of 2 groups of 3 sand bags will drive our basket 6 units up, which is corresponding of adding 6 balloons, so $-2 \cdot (-3) = 6$

Addition of 6 balloons (2 groups of three balloons) of cause will help us to move our basket up for 6 units. If we remove 2 groups of 3 balloons, we will descend 6 units.

$$-2 \cdot (+3) = -6$$

When we write a positive number, like (+6), we are usually omitting "+" sign, (+6) is the same as just 6. The expression $(+2) + (+6)$ means that we are adding two positive numbers, 2 and 6.

$$(+2) + (+6) = 2 + 6$$

When we write negative numbers, we have to be more careful; if the negative number is the first in the expression, it can be written as is:

$$(-2) + 6 = -2 + 6$$

If the negative number is not the first, it's a good idea to put it into parenthesis:

$6 + (-2)$. Expression written as $6 + -2$ looks awkward. Also, we know that the addition of a negative number is the same as subtraction of the positive one, opposite to negative $6 + (-2) = 6 - 2$.

If we want to subtract a positive number, we can write $6 - (+2)$, but the plus sign can be omitted: $6 - 2$. If we want to subtract a negative number, we have to add the number that is positive and opposite to negative.

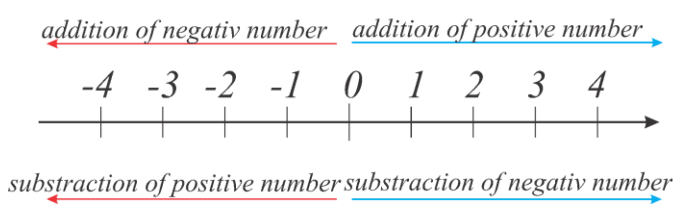
$$6 - (-2) = 6 + 2$$

Two numbers which sum is 0, are called **opposite** numbers.

6 and -6 are opposite numbers, $6 + (-6) = 0$. If we write "minus" sign in front of a number, we will get an opposite number:

$$-(-5) = 5; \qquad 5 + (-5) = 0$$

Natural numbers, their opposites, and 0 are called whole numbers or integers.



Exercises:

1. 2Write all integers

$a. from\ (-5) to\ 5; \qquad b.\ \ from\ (-4) to\ 7; \qquad c. from (-6)\ to\ 0.$

2. Write all negative whole numbers which are
   a. Greater than -8;
   b. Greater than -12 but smaller than -9;
   c. Smaller than 3 but greater than -11;

3. Fill up the table:

a	7	-4			5		0	
$-a$			0	-1		8		-3

4. Write a number, equal to
   Example: $-(-3) = 3$

   $a.\quad -(+5);$ $\qquad b.\quad +(+5);$ $\qquad c.\quad -(+5);$

   d. $-\left(-(+5)\right);$ $\qquad e.\quad -\left(-(-5)\right);$ $\qquad f.\quad -\left(-\left(-(-5)\right)\right)$

   $g.\quad \underbrace{-\left(-\left(-\left(\dots(+5)\right)\right)\right)}_{10\ "-"signs};$ $\qquad h.\quad \underbrace{\left(-\left(-\left(\dots(-5)\right)\right)\right)}_{10\ "-"signs};$

   $i.\quad \underbrace{-\left(-\left(-\left(\dots(+5)\right)\right)\right)}_{11\ "-"signs}$

5. Evaluate:

   $a.\, -1 \cdot 5;$ $\qquad b.\ 33 \cdot (-1);$ $\qquad c.\, -101 \cdot 0;$

   $d.\, -15 \cdot (-1);$ $\qquad e.\ 0 \cdot (-111);$ $\qquad f.\, -1 \cdot 0;$

6. Evaluate:

   $a.\ (+6) \cdot (-4);$ $\qquad b.\ (+15) \cdot (-2);$ $\qquad c.\ (-12) \cdot 3;$

   $d.\ (-10) : (+2);$ $\qquad e.\ (15) : (-3);$ $\qquad f.\ (-1) \cdot (+1001);$

7. Evaluate:

   $a.\ (-8) \cdot (-3);$ $\qquad b.\ (-11) \cdot (-4);$ $\qquad c.\ (-5) \cdot (-10);$

   $d.\ (-15) : (-3);$ $\qquad e.\ (-12) : (-4);$ $\qquad f.\ \left(-\frac{1}{5}\right) \cdot (-10);$

8. Find the unknown factor:

   $a.\ 345 \cdot x = -345;$ $\qquad b.\ a \cdot (-512) = -512;$

   $c.\ x \cdot (-178) = 0;$ $\qquad d.\, -421 \cdot b = 421$

9. Find the unknown factor

   $a.\, -10 \cdot x = 70;$ $\qquad b.\ -4 \cdot y = -16;$

   $c.\ x \cdot (-12) = 36;$ $\qquad d.\ y \cdot (-3) = -21$

10. Fill the table:

a	-1	4	10	-8	-4
b	1	-2	2	5	-3
c	3	-6	-5	-6	-2
$a \cdot b \cdot c$					
$(-a) \cdot b \cdot c$					
$(-a) \cdot (-b) \cdot c$					
$(-a) \cdot (-b) \cdot (-c)$					

11. Positive number, negative number or 0 is the result of the products:

$a.$ $(-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot (-5) \cdot (-6) \cdot (-7) \cdot (-8) \cdot (-9);$

$b.$ $(-1) \cdot (-2) \cdot (-3) \cdot (-4) \cdot (-5) \cdot (-6) \cdot (-7) \cdot (-8);$

$c.$ $(-3) \cdot (-2) \cdot (-1) \cdot 0 \cdot 1 \cdot 2 \cdot 3;$

$d.$ $1 \cdot (-1) \cdot 2 \cdot (-2) \cdot 3 \cdot (-3) \cdot \ldots \cdot 10 \cdot (-10);$

12. Evaluate:

$a.$ $-20 : 4 - 9;$ $b.$ $5 + 11 \cdot (-2);$ $c.$ $(-7 + 5) : (-1);$

$d.$ $-27 : (-3) - 10;$ $e.$ $0 - (-28) : (-7);$ $c.$ $-36 : (-8 + 20);$

13. Rewrite without parenthesis:

Example:

$$30 - (2 - 1) = 30 - 2 + 1$$

To check your solution, you can find the value for both part of the equality:

$30 - (2 - 1) = 30 - 2 + 1;$

$$30 - (2 - 1) = 30 - 1 = 29, \quad 30 - 2 + 1 = 29$$

$a.$ $20 + (2 - 3);$ $b.$ $20 - (2 - 3);$

$c.$ $20 - (-2 + 3);$ $d.$ $20 - \big(-2 + (-3)\big);$

14. Compare:

$-4 \ldots 4$ $\quad 6 \ldots -4$ $\quad \frac{2}{3} \ldots -\frac{3}{2}$

$-4 \ldots -2$ $\quad -4 \ldots 0$ $\quad -\frac{2}{3} \ldots -1$

$-4 \ldots -6$ $\quad -1 \ldots -\frac{1}{2}$ $\quad -2 \ldots \frac{1}{2}$

15. Evaluate:

$3+(-2);$ $3+(+2);$ $-3-(-2);$

$3-(+2);$ $-3+(-2);$ $-3+(+2);$

$3-(-2);$ $-3-(+2);$ $-3+(+3);$

16. Compare without doing actual calculation.

$a.\quad 100-(35-20) \qquad 100-(35+20)$

$b.\quad 100+(35-20) \qquad 100+(35+20)$

$c.\quad 100-(-35-20) \qquad 100-(-35+20)$

$d.\quad 100+(-35-20) \qquad 100+(-35+20)$

17. Positive or negative number will be the product of
    a. Two negative and one positive numbers.
    b. One negative and two positive numbers
    c. Three negative numbers.
    d. Three positive numbers.

18. Some digits in the numbers were changed for asterisks. Can you compare numbers?

$a.\quad ** \ldots (-1\,**);$ $\qquad b. -8\,** \ldots -9\,**;$ $\qquad c. -7\,**\,8\,* \ldots 7\,**\,2\,*$

19. Using the distributive property, represent the numerator of each fraction as a product and reduce the fractions:

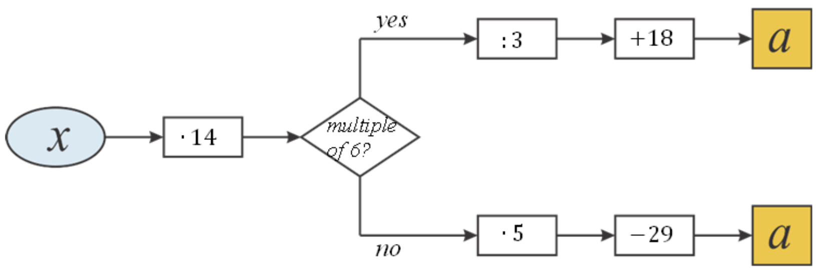
Example:

$$\frac{32\cdot5+32\cdot9}{160\cdot28}=\frac{32\cdot(5+9)}{1602\cdot28}=\frac{32\cdot14}{160\cdot28}=\frac{32\cdot14}{32\cdot5\cdot14\cdot2}=\frac{1}{5\cdot2}=\frac{1}{10}$$

$$a.\ \frac{15\cdot9-15\cdot6}{9\cdot30};\qquad b.\ \frac{17\cdot4+17\cdot9}{34\cdot52};$$

$$c.\ \frac{18\cdot7+18\cdot3}{1200};\qquad d.\ \frac{24\cdot11-24\cdot3}{300}$$

20. Using the algorithm fill the table:



x	1	2	3	4	5	6	7	8	9	10
a										

21. Victor had 80 dollars more than Mary. After Victor spent half of his money, he had 10 dollars less than Mary. How much money did Mary and Victor have together initially?

22. A problem from the "Arithmetic" of the famous Central Asian mathematician Muhammad ibn Musa al-Khwarizmi (9th century AD).
"Find the number, knowing that if you subtract one-third and one-fourth (of this number) from it, you get 10."

23. What sign should replace the asterisks, and what digit should be filled in to make the following expressions true equalities?

$a.\ 87 * 29 = 5\ ;$ $b.\ 2:4 * 77 = 1\ ;$ $c.\ 18 * 3 \cdot 9 =\ 5;$

$d.\ (96 * 48):8 = ;$ $e.\ 3 * 5 - 73 = 14\ ;$ $f.\ 300 - (80 * 3) * 6 = 2\ \ 0;$

24. Five friends were sledding down a hill. John went farther than Mike, but not as far as Robert. Dennis went a shorter distance than Mike, while Peter went farther than Robert. Who among the boys went the farthest, and who went the shortest distance?

25. Find the number, if

$a.\ \frac{2}{3}\ of\ this\ number\ is\ 6;$ $b.\ \frac{2}{5}\ of\ this\ number\ is\ 22;$ $c.\ \frac{3}{7}\ of\ this\ number\ is\ 12;$

26. Find

$a.\ \frac{2}{3}\ of\ number\ 6;$ $b.\ \frac{2}{5}\ of\ number\ is\ 25;$ $c.\ \frac{3}{7}\ of\ number\ is\ 49;$

27. Draw a number line in your notebook. Mark the fractions $\frac{1}{18}; \frac{1}{9}; \frac{1}{6}; \frac{1}{3}; \frac{1}{2}$. How many cells the unit segment of the scale should be to conveniently mark all these fractions on a same number line?

28. Compare fractions:

$a.\ \frac{32}{65} \ldots \frac{49}{65};$ $b.\frac{7}{96} \ldots \frac{7}{12};$ $c.\ \frac{14}{23} \ldots \frac{14}{37};$ $d.\ \frac{18}{19} \ldots \frac{16}{19};$

29. Find the unknown:

$a.\ x + \frac{5}{36} = \frac{13}{36};$ $b. y - \frac{16}{49} = \frac{27}{49};$

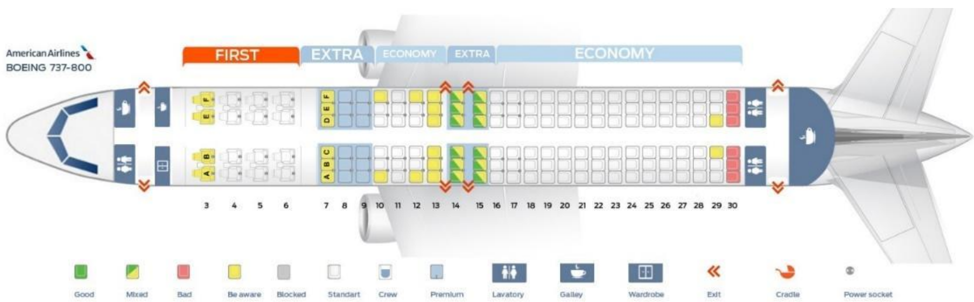
$c.\ \frac{8}{21} + k = \frac{17}{21};$ $d.\ \frac{48}{56} - t = \frac{39}{56};$

30. To which natural numbers the following fractions are equal to?

Example: $\frac{12}{3} = 4$

$a.\ \frac{16}{8};$ $b.\ \frac{18}{2};$ $c.\ \frac{24}{6};$ $d.\ \frac{30}{3};$ $e.\ \frac{35}{35};$ $f.\ \frac{51}{17}$

Chapter 8.
Coordinates.

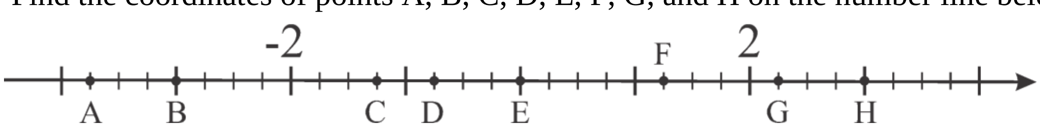


Coordinates are a set of values that show an exact position. How many values do we need to show the exact position of a point on a number line? How many values do we need to find our place in a theater? In a plane? What we can use as values?

For example, the Johns family lives in Big Village, on Main Street, house number 33, NY, USA. To describe the location of Johns' house, we use several pieces of information, such as:
Country: USA, State: NY, village: Big, Street: Main, House: 33.
On a number line, each point represents a number, and each number is linked to a point if an origin (the point at 0), a unit segment, and the positive direction are defined or can be defined based on the known information. This number is the coordinate of a point on the line in the defined system. The absolute value of this number tells us how many unit segments lie between this point and the origin, while the sign indicates on which side of the origin this point is located.

Example 1.

Find the coordinates of points A, B, C, D, E, F, G, and H on the number line below:

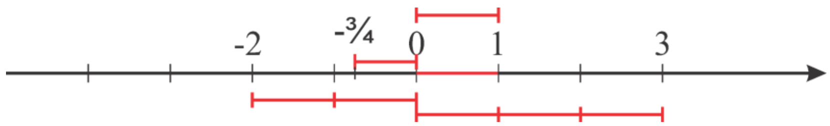


To find the coordinate of a point, we need to find how many unit segments can fit into the distance between 0 and the point, and then apply the corresponding sign, either plus or minus.
On this number line there are two known coordinates, -2 and 2. Exactly 4 unit segments fit between two points, with 0 positioned exactly in the middle. The coordinate of point E is 0, E(0). The coordinate of the point F is 1 and $\frac{1}{4}$, F$\left(1\frac{1}{4}\right)$.
Each number has a property "absolute value", it shows how far this number is from 0, the origin of the coordinate line. The formal definition of absolute value of any number is:

$$\begin{cases} |a| = a, & if\ a \geq 0 \\ |a| = -a, & if\ a < 0 \end{cases}$$

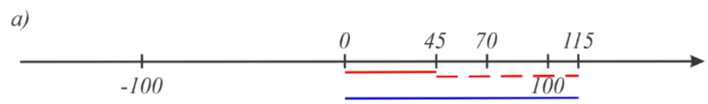
Absolute value of 2 is 2, absolute value of (-2) is also 2,

$$|2| = 2; \qquad |-2| = 2$$

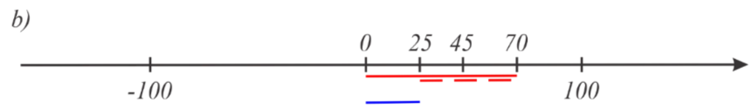


The addition and subtraction of positive and negative numbers along the coordinate axis.

Two segments represent the absolute values of numbers 45 and 70. When we need to add 45 and 70, we can sum their absolute values and the result will be positive. Addition of two positive numbers produces the positive result.
The blue segment represents the absolute value of the result.

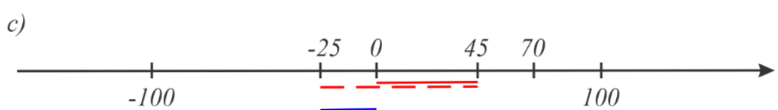


$$45 + 70 = 70 + 45 = 70 - (-45) = 45 - (-70) = 115$$



The result of subtraction of 45 from 70 is 25. The blue line represents the absolute value of the resalt, segment with the length equal to the difference of two other segments.

$$70 - 45 = 70 + (-45) = 25$$



The result of the subtraction of 70 from 45 will be the opposite of the result of $70 - 45$. The absolute value of the result will be the same (the difference between 70 and 45) but the sign will be " $-$ "

$$45 - 70 = 45 + (-70) = -(|70| - |45|) = -25$$

So, if you need to subtract one number from the other: subtract the smaller number from the greater number and decide about the resulting sign.

Example 1:

$$25 - 77;$$

25 is smaller than 77, so the answer should be negative, the absolute value of the difference is the same as the absolute value of the difference of $77 - 25$; in other words, the result is a number, opposite to the difference $77 - 25$.

$$25 - 77 = -(77 - 25) = -(+52) = -52$$

Example 2:

$$168 - 230;$$

$168 < 230$, we can find the difference $230 - 168 = 62$ and write " $-$ ".

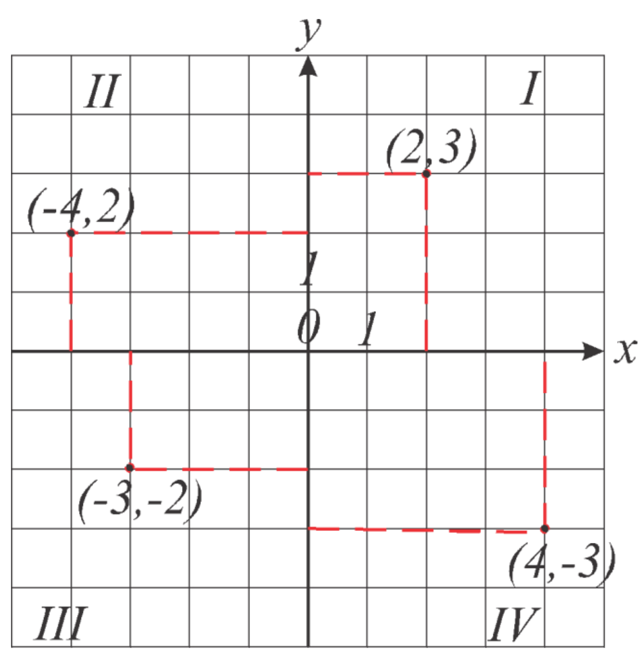
$$168 - 230 = -(230 - 168) = -62$$

Let's take a look on the equation like

$$|x| = 12$$

To which number x should be equal to make this equation true?

We are looking for a number which absolute value is 12. There are two such numbers, 12 and -12. On a plane, each point corresponds to a unique ordered pair of numbers. To define these pairs, 2 perpendicular number lines are usually used. These two number lines intersect at the point called origin, associated with pair (0,0), have the same unit segment, and are called axis, usually x and y axis.

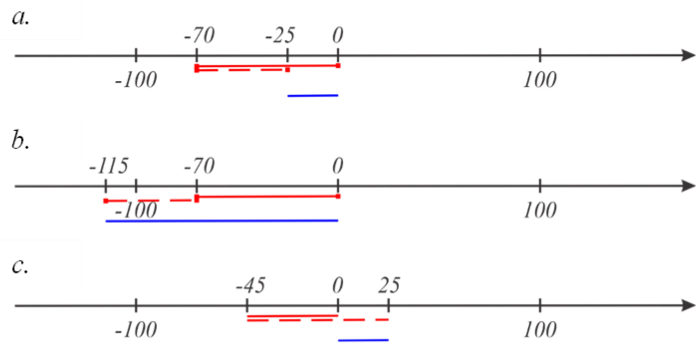


In this particular coordinate system, the two numbers allied with each point of the plane describe the distance from the point to both axes, and the signs of these numbers represent a quadrant where the point lies (quadrants I, II, III, and IV in the image above). Such a pair of numbers is an ordered pair, so the pair (n, m) and the pair (m, n) are linked to 2 different points. The absolute value of the first number in the pair is the distance to the y axis. The absolute value of the second one is the distance to the x axis.

Exercises:

1. Mark the points A(0), B(1), C$\left(-1\frac{1}{2}\right)$, D$(5)$, E$(-5)$, F$(-3)$, G$(3)$ on the number line in your notebook.

2. Write a problem, represented by the drawing:



3. Evaluate: Example: $a.\quad 7+(-2)=5$

$a.\ (-10)+(+11);$ $b.(-7)+(-6);$ $c.\ (-4)+(+2);$

$d.\ (-12)+(+3);$ $e.15-18;$ $f.\ (-11)+(-20);$

$g.\ 20-21;$ $h.(-100)+(-150);$ $i.\ (-3)+(+4);$

4. Evaluate:

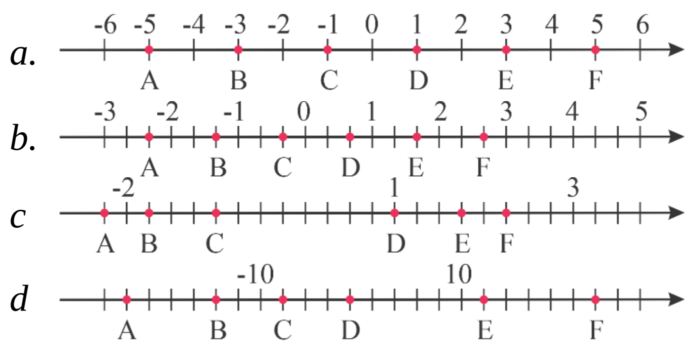
$a.\ 18-(-5);$ $b.\ -21-(-20);$ $c.\ 46-(-6);$ $d.-30-(-30);$

$e.\ 15-(-20);$ $f.\ -50-(-5);$ $g.\ 23-(-28);$ $h.-31-(-62);$

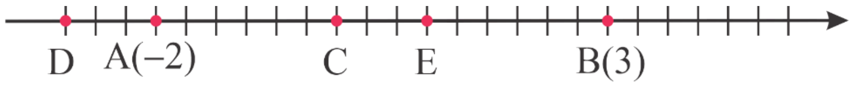
5. Fill the table:

a	20	-15	30	-10	-7
b	7	8	-9	-6	9
$a-b$					
$b-a$					

6. Write the coordinates of the points A, B, C, D, E, and F for the axes below:



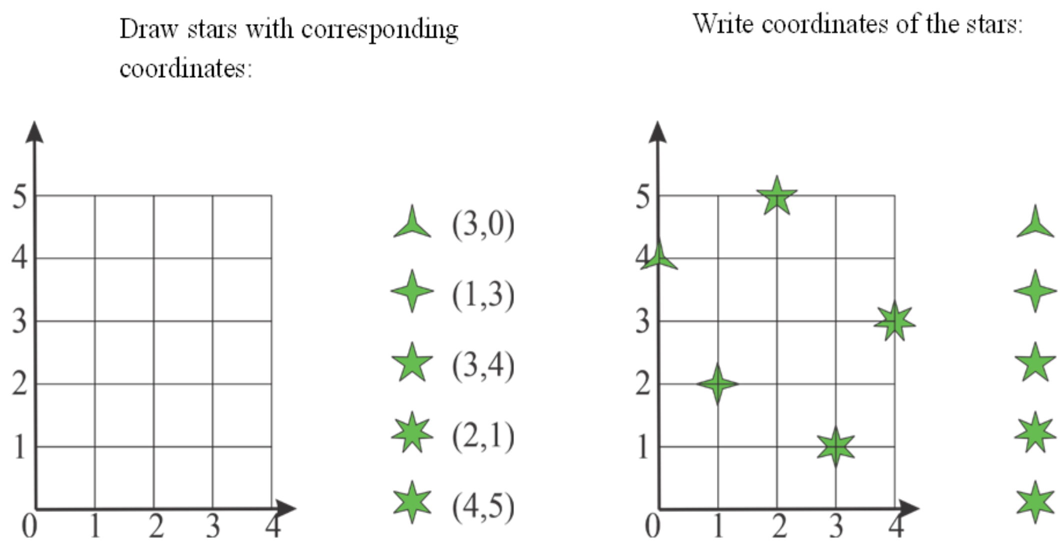
7. On the axis points $A(-2)$ and B(3) marked. Mark the origin (point is coordinate equal to 0) and unit segment. Find the coordinates of the points C, D, E.



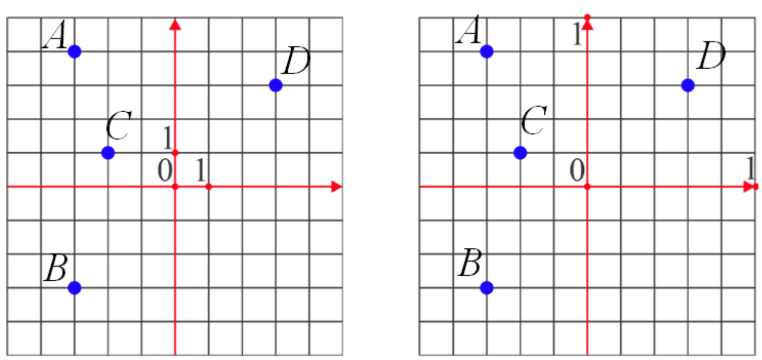
8. In your notebook, on the number line mark the points which absolute value is

$a.\ |3|;\qquad b.\ |5|;\qquad c.\ |7|;\qquad d.\,|0|.$

9. Draw the pictures in your notebook.



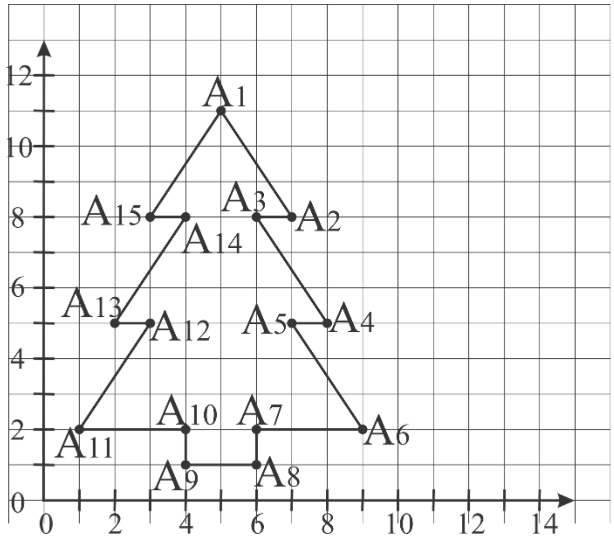
10. Find coordinates of points A, B, C, D in two different coordinate systems:



11. Draw a coordinate system (only positive quadrant I). Create a picture by coordinates (connect points in the order):

C_1 (2, 0), C_2 (2, 10), C_3 (4, 12), C_4 (12, 12), C_5 (18, 14), C_6 (18, 16), C_7 (20, 14), C_8 (22, 14), C_9 (24, 12), C_{10} (24, 14), C_{11} (25, 12), C_{12} (26, 12), C_{13} (26, 14), C_{14} (28, 12), C_{15} (28, 10), C_{16} (24, 8), C_{17} (22, 8), C_{18} (18, 6), C_{19} (18, 0), C_{20} (14, 0), C_{21} (14, 4), C_{22} (6, 4), C_{23} (6, 0), C_1(2, 0).

12. Write coordinates of the points A1, A2, …, A15 from the picture:



13. It's known that $x \cdot y = 12$. What is the value of the expressions:
$a.\ x \cdot (y \cdot 5)$; $b.\ (x \cdot 2) \cdot y$; $c.\ y \cdot (x \cdot 10)$; $d. (y \cdot 2) \cdot (x \cdot 3)$;

14. Evaluate by the convenient way:
Example:
$25 \cdot 24 = 25 \cdot (4 \cdot 6) = (25 \cdot 4) \cdot 6 = 100 \cdot 6 = 600$
$75 \cdot 8 = (25 \cdot 3) \cdot (4 \cdot 2) = (25 \cdot 4) \cdot (3 \cdot 2) = 100 \cdot 6 = 600$
$a.\ 36 \cdot 25$; $b.\ 25 \cdot 12$; $c.\ 75 \cdot 24$; $d. 150 \cdot 42$;

15. Explain the way the calculations were performed in the following examples. Using the algorithm, do the calculations:
Examples:
$238 \cdot 6 = (200 + 30 + 8) \cdot 6 = 200 \cdot 6 + 30 \cdot 6 + 8 \cdot 6 = 1200 + 180 + 48 = 1428$
$97 \cdot 14 = (100 - 3) \cdot 14 = 100 \cdot 14 - 3 \cdot 14 = 1400 - 42 = 1358$
$a.\ 104 \cdot 14$; $b.\ 102 \cdot 22$; $c.\ 98 \cdot 3$; $d. 196 \cdot 15$;

16. Evaluate by the most convenient way:
Example:
$29 \cdot 25 + 15 \cdot 6 + 19 \cdot 15 = 29 \cdot 25 + 15 \cdot (6 + 19) = 29 \cdot 25 + 15 \cdot 25 = 25 \cdot (29 + 15) = 25 \cdot 44 = 25 \cdot 4 \cdot 11 = 100 \cdot 11 = 1100$

$a.\ 12 \cdot 17 + 35 \cdot 13 + 17 \cdot 23$; $b.\ 41 \cdot 80 - 25 \cdot 41 + 55 \cdot 29$;

$c.\ 26 \cdot 18 + 26 \cdot 17 + 14 \cdot 35$

17. In the number 38*6107*, replace the asterisks with digits so that the resulting number is a multiple of
a. multiple of 2;
b. multiple of 5;
c. multiple of 10;
d. multiple of 3;
e. multiple of 9;
f. multiple of 2 and 3;
g. multiple of 15;
h. multiple of 45.

18. Mary calculated that if she gives each of her guests 2 cookies, 4 cookies will remain. However, if she gives each guest 3 cookies, 2 cookies will be missing. How many guests did Mary invite?

19. Evaluate:

$$a.\ \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{3}};\qquad b.\ \frac{\frac{5}{8}}{\frac{3}{4}\cdot\frac{1}{2}};\qquad c.\ \frac{1}{\frac{1}{2}\cdot\frac{1}{6}};\qquad d.\ \frac{\frac{2}{5}\cdot\frac{3}{4}}{15}$$

20. Evaluate (answer is 4):

$$\left(4\frac{7}{9}-2\frac{5}{6}\right):1\frac{5}{9}+\frac{4}{9}\cdot 6\frac{3}{16}$$

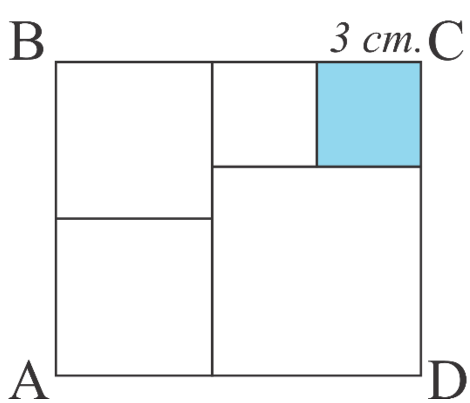
21. A student borrowed an interesting book from the library. It has 75 pages. On the first day, he read $\frac{3}{5}$ of the entire book, and on the second day, he read $\frac{2}{5}$ of the remaining pages. How many pages does he have left to read?

22. Which sign $(+, -, \cdot, \div)$ should be placed instead of * to make the following equalities true statements.

$$a.\ \frac{7}{8}*1\frac{1}{7}=1\qquad b.\ 2*1\frac{1}{3}=\frac{2}{3};\qquad c.\ \frac{3}{7}*\frac{4}{7}=\frac{3}{4};\qquad d. \frac{3}{10}*\frac{5}{6}=\frac{1}{4};$$

23. Fill the empty spaces in the magic squares with digits from 2 to 8 (from 0 to 8 for the full square) so that all the columns, rows, and diagonals have the same sum of numbers.

	0	
1		



24. The rectangle ABCD divided into 5 squares, the side of the shaded square is 3 cm. Find the sides of the rectangle.

25. Solve the following riddle (each letter represents a digit)

```
    EAT
+  THAT
  APPLE
```

26. A boy has as many brothers as sisters, but his sister has twice less sisters than brothers. How many girls and how many boys are in this family?

27. Evaluate (try to do it in a convenient way):

$$101101\cdot 555-101\cdot 555555;$$

28. In the cinema, there are two viewing halls: the red hall and the blue hall. The red hall has 40 rows, with 45 seats in each row. The blue hall has 25 rows, with 24 seats in each row. How many times more seats are there in the red hall compared to the blue hall?

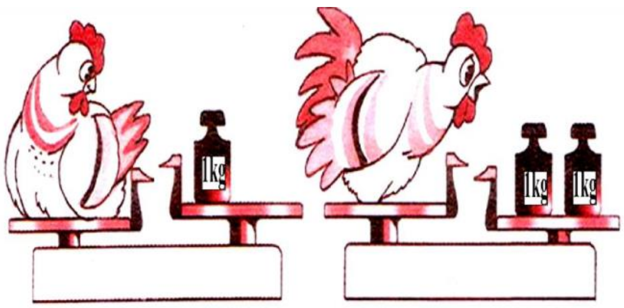
29.

a. A sequence of four numbers was written, each of which is 3 times larger than the previous one. The last number is 486. Find the first number.
b. A sequence of four numbers was written, each of which is 6 times smaller than the previous one. The last number is 2. Find the first number.

30. Find the sum $(a+b)+(a+c)+(a+a)$, if it is known that $a+b+c=8$

Chapter 9.
Decimals

In a process of measurement, we compare a standard unit, such as 1m for length, 1kg for mass, 1degree Celsius for temperature, and so on (we can use another standard units, for example 1 foot, 1 degree Fahrenheit) with the quantity we are measuring. It is very likely that our measurement will not be exact and whole

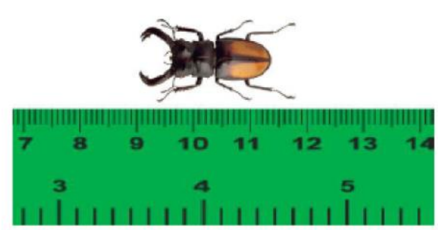


number of standard units will be either smaller, or greater than the measured quantity. In order to carry out more accurate measurement, we have to break our standard unit into smaller equal parts. We can do this in many different ways. For example, we can take $\frac{1}{2}$ of a standard unit and continue measuring. If we didn't get exact n units plus $\frac{1}{2}$ of a unit, we have to subdivide further:

$$n+\frac{1}{2}+\frac{1}{2}\cdot\left(\frac{1}{2}\right)+\cdots$$

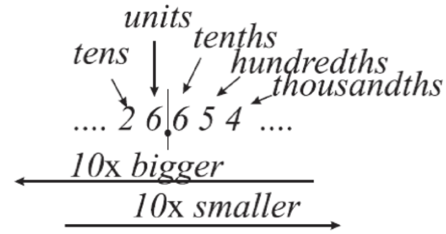
It turns out that perhaps the most convenient way is to divide a unit into 10 equal parts, then each of one tenth into another 10 even smaller equal parts and so on. In this way we will get a series of fractions with denominators 10, 100, 1000 and so on:

$$\frac{1}{10},\frac{1}{100},\frac{1}{1000}\dots$$



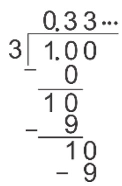
The result of our measurement can be written in a 10 based place value system.

$$
\begin{aligned}
26.654 &= 10\cdot 2 + 1\cdot 6 + \frac{1}{10}\cdot 6 + \frac{1}{100}\cdot 5 + \frac{1}{1000}\cdot 4 \\
&= 10\cdot 2 + 1\cdot 6 + \frac{6}{10}\cdot + \frac{5}{100} + \frac{4}{1000} \\
&= 10\cdot 2 + 1\cdot 6 + \frac{600}{1000} + \frac{50}{1000} + \frac{4}{1000}
\end{aligned}
$$



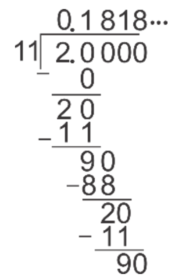
Of course, all such numbers can be expressed in the fractional notation as fractions with denominators 10, 100, 1000 …, but in decimal notation all arithmetic operations are much easier to perform.

How the fraction can be represented as decimal? One way to do it, just divide numerator by denominator, as usual. For example:



$$\frac{1}{3} = 1:3 = 0.3333\ldots = 0.\underline{3}$$

Another example,



$$\frac{2}{11} = 2:11 = 0.1818\ldots = 0.\underline{18}$$

$$\frac{3}{5} = 3:5 = 0.6$$

Can you notice the difference? If the denominator of the fraction can be prime factorized into the product of only 2 and/or 5, the fraction can be written as a fraction with denominator 10, 100, 1000 … Such fraction can be represented as a finite decimal, any other fraction will be written as infinite periodical decimal. For now, we are going to work only with finite decimals.

Examples:

$$0.3 = \frac{3}{10};\qquad 0.27 = \frac{2}{10} + \frac{7}{100} = \frac{27}{100};\qquad 0.75 = \frac{75}{100} = \frac{3\cdot 25}{4\cdot 25} = \frac{3}{4}$$

$$\frac{1}{25} = \frac{1}{5\cdot 5} = \frac{1\cdot 2\cdot 2}{5\cdot 5\cdot 2\cdot 2} = \frac{4}{10\cdot 10} = \frac{4}{100} = 0.04$$

$$\frac{7}{8} = \frac{7}{2\cdot 2\cdot 2} = \frac{7\cdot 5\cdot 5\cdot 5}{2\cdot 2\cdot 2\cdot 5\cdot 5\cdot 5} = \frac{875}{1000} = 0.875$$

As you probably already noticed, the decimal (based on 10) system of writing numbers is very consistent, and we can write very big and very small numbers in a very similar way. This system is very convenient when we perform the arithmetic calculations. Addition and subtraction can be completed by exactly the same way as with natural numbers, decimal point should be placed one on the top of the other.

$$
\begin{array}{r}
{\scriptstyle 1\,1} \\
627.402 \\
+\,164.97 \\
\hline
792.372
\end{array}
$$

How do we perform the multiplication? First, let’s see why we need to write 0 on the right side of the number when multiplying it by 10:

$$
\begin{aligned}
245\cdot 10 &= (100\cdot 2 + 10\cdot 4 + 5)\cdot 10 = 100\cdot 10\cdot 2 + 10\cdot 10\cdot 4 + 10\cdot 5 \\
&= 1000\cdot 2 + 100\cdot 4 + 10\cdot 5 + 1\cdot 0 = 2450
\end{aligned}
$$

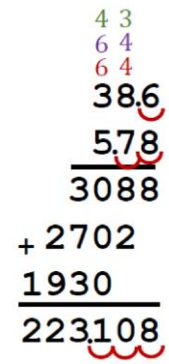
By multiplying the number by 10 we changed the place values for all digits and added one more place for units.

$$245 \cdot 100 = (100 \cdot 2 + 10 \cdot 4 + 5) \cdot 100 = 100 \cdot 100 \cdot 2 + 10 \cdot 100 \cdot 4 + 100 \cdot 5$$
$$= 10000 \cdot 2 + 1000 \cdot 4 + 100 \cdot 5 + 10 \cdot 0 + 1 \cdot 0 = 24500$$

If we need to multiply the decimal by 10 (or 100)

Using the distributive property, we proved that the result will be the number with decimal point moved one step to the right. (2 steps for multiplication by 100, and so on), It’s equivalent to increasing all place values 10 times.

Dividing by 10, we're just multiplying by $\frac{1}{10}$.



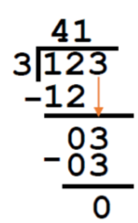
$$230{:}\,10 = 230 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 0) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{0}{10} = 20 + 3$$
$$= 23$$

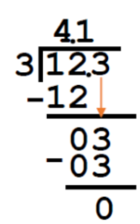
$$235{:}\,10 = 235 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 5) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{1}{10} \cdot 5$$
$$= 20 + 3 + \frac{5}{10} = 23.5$$

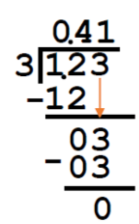
As the result, we are reducing values of each place 10 times, and we have to move the decimal point one step to the left.

To perform the long multiplication of the decimals, we do the multiplication procedure as we would do with natural numbers, regardless the position of decimal points, then the decimal point should be placed on the resulting line as many steps from the right side as the sum of decimal digits of both numbers. When we did the multiplication, we didn’t take into the consideration the fact, that we are working with decimals, it is equivalent to the multiplication of each number by 10 or 100 or 1000 … (depends on how many decimal digits it has). So, the result we got is greater by $10 \cdot 100 = 1000$ (in our example) time than the one we are looking for:

$$38.6 \cdot 5.78 = 38.6 \cdot 10 \cdot 5.78 \cdot 100{:}\,(10 \cdot 100) = 386 \cdot 578{:}\,1000$$







Division. To do the long division of the decimal by a hole number, do it as usual, but when the first digit after the decimal point is going to be placed down, put the decimal point into the answer.

If the divisor is decimal as well, it needs to be multiplied by 10(100, 1000…), depends on how many digits it has after decimal point; the dividend also should be multiplied by the same number to get the correct answer. (If we are dividing by a number which is 10 times larger, the number to be divided is also should be 10 times bigger. For example, to divide 123.452 by 1.23 we need to multiply 1.23 by 100:

$$1.23 \cdot 100 = 123$$

to get a whole number. Then multiply123.452 by 100 too.

$$123.452 \cdot 100 = 12345.2$$

Examples:

Fraction to decimals:

$$\frac{2}{10} = 0.2; \qquad \frac{3}{8} = \frac{3}{2 \cdot 2 \cdot 2} = \frac{3 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{3 \cdot 125}{10 \cdot 10 \cdot 10} = \frac{375}{1000} = 0.375;$$

$$\frac{7}{25} = \frac{7 \cdot 4}{25 \cdot 4} = \frac{28}{100} = 0.25;$$

Which fractions can be represented as a finite decimal?

$$\frac{3}{4} = \frac{3}{2 \cdot 2}\ can\ be\ represented;$$

$$\frac{5}{6} = \frac{5}{3 \cdot 2} - can't\ be\ represented, has\ 3\ as\ a\ factor.$$

Exercises:

1. Write in decimal notation the following fractions:

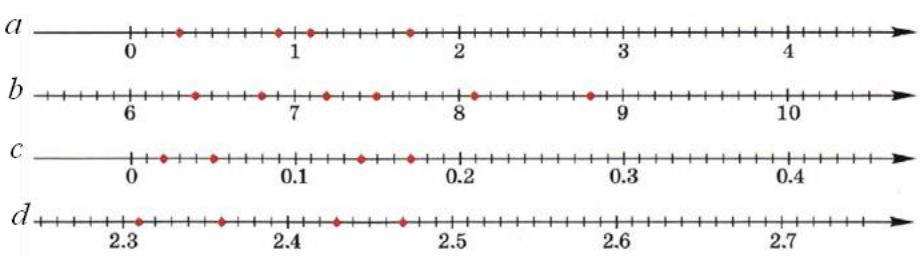
   Example:

$$1\frac{3}{25} = 1 + \frac{3}{25} = 1 + \frac{3 \cdot 4}{25 \cdot 4} = 1 + \frac{12}{100} = 1.12$$

$$a.\ 1\frac{1}{10}; \quad 2\frac{4}{10}; \quad 4\frac{9}{10}; \quad 24\frac{25}{100}; \quad 98\frac{3}{100};$$

$$b.\ 1\frac{1}{100}; \quad 4\frac{333}{1000}; \quad 8\frac{45}{1000}; \quad 75\frac{8}{10000}; \quad 9\frac{565}{10000}$$

2. Which numbers are marked on the number lines:



3. Evaluate:

$$a.\ 1.2 + 2.3 + 3.4 + 4.5 + 5.6 + 6.7 + 7.8;$$

$$b.\ 2.3 + 3.4 + 4.5 - 5.6 + 6.7 + 7.8 + 8.5 + 9.2;$$

$$c.\ 1.7 + 3.3 + 7.72 + 3.28 + 1.11 + 8.89;$$

$$d.\ 18.8 + 19 + 12.2 + 11.4 + 0.6 + 11;$$

4. On a graph paper draw a number line, use 10 squares as a unit. Mark points with coordinates 0.1, 0.5, 0.7, 1.2, 1.3, 1.9.

5. Write the numbers in an extended form;

   Example:

$$312.23 = 100 \cdot 3 + 10 \cdot 1 + 1 \cdot 2 + \frac{1}{10} \cdot 2 + \frac{1}{100} \cdot 3$$

34.2; 231.51; 76.243; 25.34; 0.23; 0.0023

6. Write in decimal notation:

$$\frac{173}{10};\quad \frac{173}{100};\quad \frac{173}{1000};\quad \frac{173}{1000};$$

7. Write in decimal notation:

$a.\ 2\frac{18}{100};\quad 5\frac{3}{100};\quad 1\frac{238}{1000};\quad 8\frac{8}{1000};\qquad b.\ \frac{39}{10};\quad \frac{187}{10};\quad \frac{341}{100};\quad \frac{1002}{1000}$

8. Which fractions below can be written in as a finite decimal:

$$\frac{1}{2},\quad \frac{2}{3},\quad \frac{1}{4},\quad \frac{1}{5},\quad \frac{1}{6},\quad \frac{1}{7},\quad \frac{1}{8},\quad \frac{1}{9},\quad \frac{1}{10},$$

$$\frac{1}{11},\quad \frac{1}{12},\quad \frac{1}{13},\quad \frac{1}{14},\quad \frac{1}{15},\quad \frac{1}{16},\quad \frac{1}{17}$$

Why do you think so?

9. Write decimals as fractions and evaluate the following expressions:

$a.\ \frac{2}{3}+0.5;\qquad b.\ \frac{1}{3}\cdot 0.9;\qquad c.\ \frac{3}{16}\cdot 0.16$

$d.\ 0.6-\frac{2;}{5}\qquad e.\ 0.4:\frac{2}{7};\qquad f.\ \frac{9}{20}:0.03$

10. Without performing calculations, for each expression from the first row, find the corresponding equal expression from the second row and write the corresponding equalities.

Example:

$$0.125+\frac{1}{4}=\frac{1}{8}+0.25$$

$\frac{3}{4}-0.5;\qquad \frac{1}{4}-0.2;\qquad \frac{1}{2}-0.125$

$0.5-\frac{1}{8};\qquad 0.75-\frac{1}{2};\qquad 0.25-\frac{1}{5};$

11. Answer:

Which part of 1 m is 1 cm? Which part of 1 km is 1 m?
Which part of 1 cm is 1 mm? Which part of 1 m is 1 dm?
Which part of 1 kg is 1 g? Which part of 1 g is 1 mg?

12. 1 kilogram of candies costs 16 dollars. How much

a. 0.5 kg will cost?
b. 1.2 kg will cost?
c. 0.75 kg will cost?
d. 0.4 kg will cost?
e. 2.5 kg will cost?

13. Represent time in hours, if possible, write the answer as a finite decimal.

$a.\ 1\ h\ 12\ min;\qquad b.\ 2\ h\ 20\ min;\qquad c.\ 10\ h\ \ 45\ min;$

$d.\ 1\ h\ 40\ min;\qquad e.\ 3\ h\ 50\ min;\qquad f.\ 2\ h\ \ 48\ min;$

14. What digit can be placed instead of asterisks, so the expressions are true?

    $a.\ 0.488 < 0.4*8;$ $b.\ 1*93 < 11.93;$ $c.\ 3.07 < 3.0*;$ $d.\ 6.*9 < 6.38$

15. Find the unknown:

    $a.\ 3.3 - 0.3y = 0.33$ $b.\ b:8 - 0.88 = 8.8$

16. In a driving school, a car with an instructor and three students went for a ride. The instructor drove $\frac{2}{5}$ of the total distance plus 5 km, two students each drove $\frac{1}{4}$ of the distance, and the third student drove the remaining 10.5 km. What was the total length of the itinerary?

17. Ancient Greek scientist Aristotle was born in 384 and died in 322. Another Greek scientist, Pythagoras, was born in 570 and died in the year 495. Ancient Greek historian Plutarch was born in 46 and died in 120. Who among them was born earlier? For how long did they live?

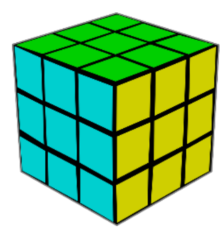
18. Evaluate (answer is 25):

$$\frac{5.6\cdot 3\frac{1}{3}\cdot 0.63}{4.9\cdot 0.018\cdot 5\frac{1}{3}};$$

19. A few kids went to the forest to pick mushrooms. If Anna gives half of her mushrooms to Vita, all the children will have the equal number of mushrooms, if instead Anna gives all her mushrooms to Alex, then Alex will have as many mushrooms as all the other kids combined. How many kids went to the forest for mushrooms?

20. On Halloween night, Peter ate half of the chocolates he had collected. The next day, he ate half of the remaining candies and gave the rest to his younger brother. He gave his brother 5 chocolates. How many candies did Peter collect?

21. A cube is cut into 27 identical smaller cubes by making two cuts parallel to each of the three pairs of cube's faces (similar to Rubik's cube).



    a. How many small cubes will have three faces painted?
    b. How many small cubes will have two faces panted?
    c. How many small cubes will have one face painted?
    d. How many small cubes will not have painted faces at all?

22. The farmer brought a basket of apples to the market. To the first customer, he sold half of all his apples and half an apple more, to the second customer - half of the remainder and half an apple more, to the third - half of the remainder and half an apple more, and so on. However, when the sixth customer came and bought half of the remaining apples and half an apple, it turned out that, like the other buyers, all his apples were whole, and the farmer sold all his apples. How many apples did he bring to the market?

23. Evaluate:

$$\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{13\ times}+\underbrace{\frac{1}{4}+\frac{1}{4}+\cdots+\frac{1}{4}}_{7\ times}-\underbrace{\frac{1}{4}+\frac{1}{4}+\cdots+\frac{1}{4}}_{25\ times};$$

24. How will the product change if:
    a. one factor is increased 9 times;
    b. one factor is decreased 7 times;
    c. one factor is decreased 2 times, and the other is decreased 8 times;
    d. one factor is increased 4 times, and the other is increased 5 times;
    e. one factor is increased 12 times, and the other is decreased 4 times;
    f. one factor is increased 3 times, and the other is decreased 6 times;
    g. one factor is increased n times, and the other is increased 2 times;
    h. one factor is decreased t times, and the other is decreased 3 times?

25. *252 students from the school are going on a field trip. Several identical buses are ordered for them. However, it turned out that if buses with 6 more seats were ordered, one less bus would be needed. How many larger buses need to be ordered if, in both cases, all buses are expected to be filled with no empty seats?

26. There are the same number of apples in each of the five boxes. If 60 apples are removed from each box, the total number of apples left in all the boxes will be equal to the number of apples that were originally in two boxes. How many apples were there in each box?

27. The sum of six different natural numbers is 22. Find these numbers.