MATH 10
ASSIGNMENT 21: CONTINUOUS FUNCTIONS AND INTERMEDIATE VALUE THEOREM

MARCH 23, 2025

Definition. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called continuous if, for every sequence $a_n \in \mathbb{R}$ which has a limit: $\lim a_n = A \in \mathbb{R}$, the sequence $f(a_n)$ also has a limit and $\lim f(a_n) = f(A)$.

It was proved last time that the sum and product of continuous functions is continuous; the same is true for f/g as long as $g \neq 0$. In particular, all polynomials and rational fucntions are continuous everywhere they are defined.

Theorem (Intermediate Value Theorem)**.** *Let $f(x)$ be a continuous fucntion on the interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$. Then there exists a point $c \in (a, b)$ such that $f(c) = 0$.*

A proof was discussed in class.

Homework

1. Prove that polynomial $x^3 + 3x - 2$ has a root between 0 and 5.

2. Prove that there exists a positive number x such that $\sin(x) = 0.5x$. (You can use without proof the fact that $\sin(x)$ is continuous).

3. Let $f(x) = x^{2n+1} + \dots$ be a polynomial of odd degree, with leading coefficient 1.
   (a) Prove that for large enough x, $f(x) > 0$ (i.e., there exists a real number M such that for all $x \geq M$, $f(x) > 0$.)
   (b) Prove that for large enough x, $f(-x) < 0$.
   (c) Porve that $f(x)$ has at least one real root.

4. A traveler leaves town A at 9 am on Monday and arrives at town B at 4 pm the same day. He spends the night at town B, leaves it at 9 am on Tuesday, and returns to town A by 4 pm on Tuesday, following the same road.

Prove that there is a point on the road which he passed at exact same time on Monday and Tuesday.

Note that we are not assuming that the traveler goes at constant speed.

5. Recall that a polygon is called *convex* if any segment with endpoints on the boundary of that polygon, the whole segment is inside the polygon.

Given a convex polygon S and a point A inside it, prove that there exists a chord of S which has A as the midpoint. [Hint: consider difference of lengths of the two pieces of a chord through A as a function of the angle.]

6. We are given 10 marked points in the plane, such that no three of them are on the same line. Prove that there is a line such that on each side of it there are 5 marked points.

***7.** We are given 10 red and 10 blue points in the plane, such that no three of them are on the same line. Prove that there is a line such that on each side of it there are 5 red and 5 blue points.