

MATH 10
ASSIGNMENT 20: CONTINUOUS FUNCTIONS

MARCH 16, 2025

Definition. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called continuous if, for every sequence $a_n \in \mathbb{R}$ which has a limit: $\lim a_n = A \in \mathbb{R}$, the sequence $f(a_n)$ also has a limit and $\lim f(a_n) = f(A)$.

For example, function $f(x) = x$ is continuous, while the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not (see Problem 1 below)

Instead of functions $f: \mathbb{R} \rightarrow \mathbb{R}$, the same definition can be applied to functions between other sets: if X, Y are metric spaces (i.e., have sets with the notion of distance, satisfying all required properties), then the above definition works without any changes for functions $f: X \rightarrow Y$. For example, we can talk about continuous functions on an interval $[0, 1]$ or on the set \mathbb{R}_+ of positive real numbers. Note that in the latter case we only require that $f(a_n)$ converge for a sequence $a_n \in \mathbb{R}_+$ which has a limit also in \mathbb{R}_+ . For example, we do not require that $f(1/n)$ converge.

HOMEWORK

1. Prove that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not continuous, by constructing a sequence a_n which has a limit but for which $f(a_n)$ has no limit.

2. Prove that the function $f(x) = x^2 + 1$ is continuous.

3. Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions.

- (a) Prove that $f + g, f - g, fg$ are also continuous. [Hint: remember the limit laws for sequences from assignment 15?]
- (b) Prove that f/g is continuous on the set $X = \{x \in \mathbb{R} \mid g(x) \neq 0\} \subset \mathbb{R}$.
- (c) Deduce that any polynomial is continuous everywhere on \mathbb{R} , and a rational function $f(x) = p(x)/q(x)$, where $p(x), q(x)$ are polynomials, is continuous everywhere it is defined.

4. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.

- (a) Prove that then the set $A = \{x \in \mathbb{R} \mid f(x) > 0\}$ is open in \mathbb{R} (if you have forgotten the definition of open set, review Assignment 13). Hint: otherwise, set A contains a point $x \in \partial A$; then choose a sequence $x_n \in A'$ such that $\lim x_n = x$, where $A' = \{x \in \mathbb{R} \mid f(x) \leq 0\}$ is the complement of A .
- (b) Prove that the set $B = \{x \in \mathbb{R} \mid f(x) \geq 0\}$ is closed in \mathbb{R}

- *5. Modify the previous proof to show that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, and $U \subset \mathbb{R}$ is open, then $f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$ is also open.