MATH 10
ASSIGNMENT 15: LIMITS CONTINUED
JAN 19, 2025

Today we will be discussing limits of sequences of real numbers (but many of the results coudl be generalzied to sequences of points in a plane, or in fact to sequences in any metric space).

Recall the definition of limit:

Definition. A number a is called the *limit* of sequence a_n (notation: $a = \lim a_n$) if for any $\varepsilon > 0$, all terms of the sequence starting with some index N will be in the ε-neigborhood of a: for any $n \geq N, |a_n - a| < \varepsilon$.

However, when woirking with sequences of real numbers, usually one computes limits not by using this definition but rather using the following *limit laws*:

Theorem 1. *Let sequences a_n, b_n be such that $\lim a_n = A$, $\lim b_n = B$. Then:*

1. $\lim(a_n + b_n) = A + B$
2. $\lim(a_n b_n) = AB$
3. $\lim(a_n/b_n) = A/B$ *(only holds if $B \neq 0$).*

In addition, there is also the following result:

Theorem 2. *If $a_n \geq 0$, $\lim a_n = 0$, and $|b_n| \leq a_n$, then $\lim b_n = 0$.*

The following limits are useful:

- If a_n is a constant sequence: $a_n = c$ for all n, then $\lim a_n = c$
- $\lim \frac{1}{n} = 0$
- If $|r| < 1$, then $\lim r^n = 0$

Sometimes in order to use these rules, some tricks are necessary. For example, one can not compute the limit $\lim \frac{n+2}{2n+3}$ direclty, as $\lim(n+2)$ does not exist. However, a simple trick allows one to use the quotient rule:

$$\lim \frac{n+2}{2n+3} = \lim \frac{1 + \frac{2}{n}}{2 + \frac{3}{n}} = \frac{1+0}{2+0} = \frac{1}{2}$$

Using these rules, we had computed the following important limit

$$1 + r + r^2 + \cdots = \lim(1 + r + \cdots + r^n) = \lim \frac{1 - r^n}{1 - r} = \frac{1}{1 - r}, \qquad |r| < 1 \tag{1}$$

1. Compute the following limits.
(a) $\lim \frac{2n^2+n+1}{n^2+3}$
(b) $\lim \frac{n^2+15n}{n^3}$
(c) $\lim \frac{2^n+1}{3^n}$
*(d) $\lim \frac{n}{2^n}$ [Show first that for $n \geq 3$, one has $a_{n+1}/a_n \leq 2/3$. Deduce then that $a_n \leq C(2/3)^n$ for some constant C.]

2. Prove that a sequence that has a limit must be bounded, i.e. there exists a number M such that for all indices n, we have $|a_n| < M$. [Hint: if $\lim a_n = A$, then starting from some moment, all terms of the sequence are $\leq A + 1$.]

***3.** Prove Theorem 2.

4. Prove that if $|b_n| \leq 2$, and $\lim a_n = 0$, then $\lim a_n b_n = 0$. Note that we do not assume that limit $\lim b_n$ exists.

5. Consider the sequence defined by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) \tag{2}$$

(a) Use a calculator or a computer to compute the first 5 terms. Does it indeed look like the sequence is convergent? [You are not required to give a rigorous proof that it is convergent.]

(b) Assuming that it does converge, can you guess what the value of the limit is? [Hint: if this sequence is convergent, then the limit A satisfies $A = \frac{1}{2}(A + \frac{2}{A})$.]
(c) Can you modify (2) to get a sequence that computes $\sqrt{3}$?

6. Consider the sequence given by $x_1 = 1$, $x_{n+1} = \frac{1}{1+x}$.
(a) Compute first 3 terms of this sequence.
(b) Prove that if the limit exists, it satisfies $X = \frac{1}{1+X}$.
(c) Assuming that the limit exists, find it.

7. Consider the infinite decimal

$$x = 0.17171717\ldots$$

(a) Show that this decimal can be written as a sum of an infinite geometric progression.
(b) Show that x is a rational number.
(c) Is it true that any periodic infinite decimal is rational? Is the converse true?

8. (a) Let S be a closed set and a_n a sequence such that $a_n \in S$ for any n. Prove that if the limit $\lim a_n$ exists, it must be also in S. [Hint: otherwise, the limit is in the complement S', and the complement is open...]
(b) Let $a_n \geq 0$ for all n. Prove that then $\lim a_n \geq 0$ (assuming it exists).
(c) Let $a_n > 0$ for all n. Is it true that then $\lim a_n > 0$ (assuming it exists)?