MATH 10
ASSIGNMENT 14: LIMITS
JANUARY 12, 2025

Limits

Let X be a metric space and a_n be a sequence in X. We say that a sequnce a_n has limit A if, as n increases, terms of the sequence get closer and closer to A.

This definition is not very precise. For example, the terms of sequence $a_n = 1/n$ get closer and closer to 0, so one expects that the limit is 0. On the other hand, it is also true that they get closer and closer to -1. So the words "closer and closer" is not a good way to express what we mean.

A better way to say this is as follows.

Definition. A set U is called a *trap* for the sequence an if, starting with some index N, all terms of the sequence are in this set:

$$\exists N : \forall n \geq N : a_n \in U$$

Note that it is not the same as "infinitely many terms of the sequence are in this set". Now we can give a rigorous definition of a limit.

Definition. A point $A \in X$ is called the *limit* of sequence an (notation: $A = \lim a_n$) if for any $\epsilon > 0$, the neighborhood $B_\epsilon(A) = \{x | d(x, A) < \epsilon\}$ is a trap for the sequence a_n.

For example, when we say that for a sequence of real numbers an we have $\lim a_n = 3$, it means:

there is an index N such that for all $n \geq N$ we will have $a_n \in (2.99, 3.01)$,
there is an index N' (possibly different) such that for all $n \geq N_0$ we will have $a_n \in (2.999, 3.001)$
there is an index N'' such that for all $n \geq N''$ we will have $a_n \in (3 - 0.0000001, 3 + 0.0000001)$

Problems

1. Consider the sequence $a_n = 1/n$
  (a) Fill in the blanks for each of the statements below so they become true statements:
  - For all $n \geq$ ______, $|a_n| < 0.1$
  - For all $n \geq$ ______, $|a_n| < 0.001$
  - For all $n \geq$ ______, $|a_n| < 0.0017$

  (b) Show that $\lim a_n = 0$.

2. Prove that $\lim \frac{1}{n(n+1)} = 0$ (hint: $\frac{1}{n(n+1)} < \frac{1}{n}$).

3. Find the limits of each of these sequences if they exist:
  (a) $a_n = \frac{1}{n^2}$
  (b) $a_n = \frac{1}{2^n}$
  (c) $a_n = n$

4. Explain why 1 is *not* a limit of the sequence $a_n = (-1)^n$.

5. (a) Show that if a sequence of real numbers has limit $\lim a_n = -1$, then starting from some index, all terms in the sequence are negative.
  (b) Show that it is impossible for the sequence to also have a limit $a_n = 1$.

6. (a) Show that if all terms of a sequence of real numbers are non-negative, then its limit (if exists) is also non-negative.
  (b) Is it true that if all terms of a sequence are positive, then its limit (if exists) is also positive?

7. Let $\lim a_n = A$, and let U be an open set containing A. Show that then, starting with some index N, all terms of the sequence an are in U.

8. Let $S \subset X$ be a closed set. Let a_n be a sequence of real numbers such that for all n, $a_n \in S$. Suppose also that a_n has a limit. Show that $\lim a_n \in S$. [Hint: let S' be the complement of S'. Then S' is open, so every point of S' is an interior point ...].

9. Show that the limit of a sequence, if exists, is unique: it is impossible that $\lim a_n = A$ and also $\lim a_n = A'$, with $A \neq A'$. [Hint: A, A' have non-intersecting neighborhoods.]

10. Consider a *sequence of sequences*, which $(a_n)_m$. For example, $(a_n)_1$ is one sequence labeled by n, and $(a_n)_2$ is another. We may label the limit of the sequence $(a_n)_m$ as A_n. Alternatively, we may rewrite this as a variable labeled by *two* integers, $a_{n,m}$ and consider limits with respect to either variable. So, we may write $A_m = \lim_n a_{n,m}$, giving us a new sequence A_m. The subscript n on lim denotes that we are taking the limit with respect to n. We may also consider *another* sequence of limits $B_n = \lim_m a_{n,m}$ for the *same* variables $a_{n,m}$. Find such a sequence of sequences, $a_{n,m}$, such that $\lim_m \lim_n a_{n,m}$ and $\lim_n \lim_m a_{n,m}$ both exist, but are not equal (in this case we say that the *limits don't commute*).