MATH 10
ASSIGNMENT 11: MATRICES
DEC 8, 2024

Linear Maps and Matrices

Recall that an $m \times n$ matrix is just a rectangular array of numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

We will denote set of all $m \times n$ matrices by $Mat_{m\times n}$.

We have used matrices before to describe a system of linear equations. Another, closely related use is for describing maps (functions) between spaces.

Given a matrix $A \in Mat_{m\times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$, define their product $A\mathbf{x}$ by

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$$

Note that the result is a vector with m components; thus, an $m \times n$ matrix gives a map (or a function) $\mathbb{R}^n \to \mathbb{R}^m$. It is easy to see that this map is linear:

$$A(c\mathbf{x}) = c \cdot A\mathbf{x}, \qquad c \in \mathbb{R}$$
$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$$

(in fact, any linear map $\mathbb{R}^n \to \mathbb{R}^m$ is described by a matrix).

Using this notation, we can write any system of linear equations, with m equations and n variables

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$
$$\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m$$

in a compact form:

$$A\mathbf{x} = \mathbf{b}.$$

Spaces of solutions

We say that a subset V of $\mathbb{R}^n$ is a *subspace* if for any $\mathbf{v}, \mathbf{v}' \in V$ and a real number c, we have $c\mathbf{v} \in V$ and $\mathbf{v} + \mathbf{v}' \in V$ (in particula,r it means that $0 \in V$). For example, subset of $\mathbb{R}^3$ given by equation $x_1 + x_2 = 0$ is a subspace. More generally, for any equation of the form $A\mathbf{x} = 0$, the space of solutions is a subspace of $\mathbb{R}^n$.

We say that subspace V has dimension d if one can find d vectors $\mathbf{v}_1, \dots \mathbf{v}_d$ in V such that

1. Every vector in V can be written in the form

$$\mathbf{x} = t_1\mathbf{v}_1 + \dots + t_d\mathbf{v}_d$$

for some $t_1, \dots, t_n \in \mathbb{R}$. (Expressions of this form are called *linear combinations* of vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$.)

2. Vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ are independent: no one of them can be written as a combination of others.

In this case, the collection $\mathbf{v}_1, \dots, \mathbf{v}_d$ is called a *basis* of V, and every vector can be *uniquely* written as a linear combination of basis vectors.

For example, the space $\mathbb{R}^n$ itself has dimension n: one can take the standard basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad , \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

so that

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$$

Important: the same vector space can have many different bases, and coordinates of a vector depend on the choice of basis — see problem 1 below. However, it is known that all bases have the same number of basis elements; this number is called *dimension* of the vector space.

More generally, if V is the set of vectors $\mathbf{a} + t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$, and vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are independent, we say that V has dimension d (in this sitутation, V is called an affine subspace — which just means a usual subspace but translated away from the origin).

Theorem. *The dimension of the space of solutions of a system of linear equations $A\mathbf{x} = 0$ is given by*

$$d = (\textit{number of variables}) - (\textit{number of nonzero rows in row echelon form})$$

More generally, for system of equations $A\mathbf{x} = \mathbf{b}$, if it has any solutions at all, then the dimension of the set of solutions is given by the same formula as above.

Indeed, the number of free variables is the number of all variables minus the number of pivot variables.

Thus, typically we expect that a system with n variables and k equations has $n - k$ dimensional space of solutions. This is not always true: it could happen that after bringing it to row echelon form, some rows become zero, or that the system has no solutions at all — but these situations are unusual (at least if $k \le n$).

Corollary. *For a system of equations $A\mathbf{x} = 0$, if number of equations m is strictly less than the number of variables n, then system $A\mathbf{x} = 0$ always has at least one non-zero solution.*

Homework

1. Show that vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

form a basis in $\mathbb{R}^2$: every vector $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be uniquely written in the form

$$v = x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2$$

Can you express x_1', x_2' in terms of x_1, x_2? conversely, can you express x_1, x_2 in terms of x_1', x_2'?

2. (a) Consider the following 3 vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Show that there is a linear relation between them: one can find real numebrs c_1, c_2, c_3 (not all zero) such that

$$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = 0.$$

(b) Deduce from the previous part that these 3 vectors do not form a basis: vector 0 can be written as combination of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in more than one way.

(c) Use corollary above to show that in fact, for any 3 vectors in $\mathbb{R}^2$, there must be a linear relation between them. [Hint: $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = 0$ is equivalent to a system of linear equations on c_1, c_2, c_3.]

*(d) Can you state and prove similar statement for $\mathbb{R}^n$?

3. Assume that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is a basis in $\mathbb{R}^3$ (not necessarily the standard one), with the additional property that each vector has unit length and they are all orthogonal to each other:

$$|\mathbf{e}_i| = 1, \qquad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ for } i \neq j$$

Since it is a basis, each vector can be written in the form $v = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.

Prove that then one can find the coordinates x_1, x_2, x_3 in this basis very easily:

$$x_i = v \cdot \mathbf{e}_i$$

4. (a) Let R be the operation of counterclockwise rotation by 90 degrees in the plane $\mathbb{R}^2$. Show that it can be described by a matrix:

$$R\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

for some 2×2 matrix A.

(b) Can you do the same, but for the operation R_θ of rotation by angle θ? [Hint: let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis in $\mathbb{R}^2$. Can you find $R_\theta\mathbf{e}_1$, $R_\theta\mathbf{e}_2$? Once you do that, finding $R_\theta(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$ should be easy]

(c) Can you do the same for the operation of rotation by angle θ around z-axis in $\mathbb{R}^3$? around x-axis?

5. Solve the system of linear equations

$$\begin{aligned} y - 3z &= -1 \\ x - y + 2z &= 3 \\ 2x - y + 4z &= 8 \end{aligned}$$

6. Matrix multiplication. Let A, B be $n \times n$ matrices; then each of them defines a linear map $\mathbb{R}^n \to \mathbb{R}^n$: $\mathbf{x} \mapsto A\mathbf{x}$, $\mathbf{x} \mapsto B\mathbf{x}$. Consider their composition: $\mathbf{x} \mapsto A(B\mathbf{x})$.

(a) Show that this map is linear:

$$A(B(c\mathbf{x})) = cA(B\mathbf{x}), \quad c \in \mathbb{R}$$
$$A(B(\mathbf{x}+\mathbf{y}) = A(B\mathbf{x}) + A(B\mathbf{y})$$

(b) Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Compute $A(B\mathbf{x})$, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Can you find a matrix C such that $A(B\mathbf{x}) = C(\mathbf{x})$?

(c) In general, show that for any $n \times n$ matrices A, B one can find the matrix C such that $A(B\mathbf{x}) = c\mathbf{x}$. Thius matrix is called the product of matrices A, B and written $C = AB$.

7. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Compute A^2, A^3, A^4. Can you guess and prove the general formual for A^n?