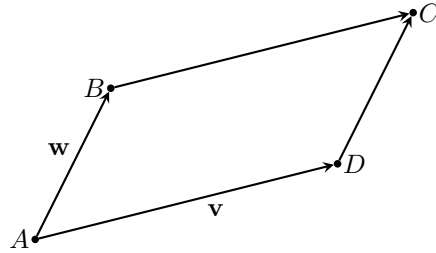


MATH 10
ASSIGNMENT 7: SIGNED AREA
 NOV 3, 2024

SIGNED AREA

Let $ABCD$ be a parallelogram on the plane, with vertex A at the origin and vertices $D = (x_1, y_1)$, $B = (x_2, y_2)$, so that its sides are vectors

$$\mathbf{v} = \vec{AD} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \mathbf{w} = \vec{AB} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$



In this case, the area of the parallelogram can be computed as follows:

$$(1) \quad S_{ABCD} = |x_1y_2 - y_1x_2|$$

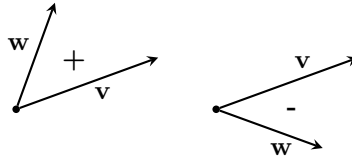
(we will prove this in problem 1 below). We will introduce a new kind of “product” for two vectors \mathbf{v}, \mathbf{w} in \mathbb{R}^2 by

$$(2) \quad \mathbf{v} \wedge \mathbf{w} = x_1y_2 - y_1x_2 \in \mathbb{R}$$

if \mathbf{v}, \mathbf{w} are as above (symbol \wedge reads “wedge”). Thus, $S_{ABCD} = |\vec{AD} \wedge \vec{AB}|$. Note that as with the dot product, the wedge product is a number, not a vector.

One can think of $\mathbf{v} \wedge \mathbf{w}$ as “signed area”:

$$\mathbf{v} \wedge \mathbf{w} = \begin{cases} S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is counterclockwise} \\ -S_{ABCD}, & \text{if rotation from } \mathbf{v} \text{ to } \mathbf{w} \text{ is clockwise} \end{cases}$$



The wedge product (and thus, the signed area) is in many ways easier than the usual area. Namely, we have:

1. It is linear: $(\mathbf{v}_1 + \mathbf{v}_2) \wedge \mathbf{w} = \mathbf{v}_1 \wedge \mathbf{w} + \mathbf{v}_2 \wedge \mathbf{w}$
2. It is anti-symmetric: $\mathbf{v} \wedge \mathbf{w} = -\mathbf{w} \wedge \mathbf{v}$

HOMEWORK

1. The goal of this problem is to give a careful proof of formula (1).
 - (a) Show that $S_{ABCD} = |\mathbf{v} \cdot R(\mathbf{w})|$, where R is the operation of rotating by 90° clockwise. [Hint: $S = |\mathbf{v}| |\mathbf{w}| \sin(\varphi)$.]
 - (b) Show that for a vector $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$, we have $R(\mathbf{w}) = \begin{bmatrix} y \\ -x \end{bmatrix}$.
 - (c) Deduce from this formula (1).
2. Let \mathbb{S}_{ABC} be the signed area of triangle ABC :

$$\mathbb{S}_{ABC} = \begin{cases} S_{ABC} & \text{if vertices } A, B, C \text{ go in counterclockwise order} \\ -S_{ABC} & \text{if vertices } A, B, C \text{ go in clockwise order} \end{cases}$$

Note that \mathbb{S}_{ABC} depends not just on the triangle but also on the order in which we list the vertices. Show that

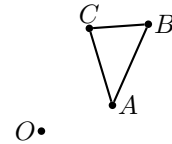
$$\mathbb{S}_{ABC} = \frac{1}{2} \vec{AB} \wedge \vec{AC}.$$

3. Find the area of the triangle with vertices at $(0, 0)$, $(5, 1)$, $(7, 7)$.
4. If the area of $\triangle ABC$ is 24, what is the area of $\triangle ABM$, where M is the intersection point of the medians?
[This problem can be solved in many ways. One of them: if $\vec{AB} = \mathbf{v}$, $\vec{AC} = \mathbf{w}$, then what is \vec{AM} ?]

5. Shoelace formula.

- (a) Consider a triangle ABC in the plane; let \mathbb{S}_{ABC} be as in problem 2. Show that then for any point O in the plane, we have

$$\mathbb{S}_{ABC} = \mathbb{S}_{OAB} + \mathbb{S}_{OBC} + \mathbb{S}_{OCA} = \mathbb{S}_{OAB} + \mathbb{S}_{OBC} - \mathbb{S}_{OAC}$$



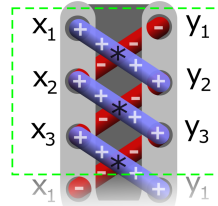
Note that there are many possible configurations: for example, O could be on the other side of BC , or it could be inside ABC . Do you think the above formula holds in all configurations or only in some?

- (b) Consider triangle ABC , where $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$. Show that then,

$$\mathbb{S}_{ABC} = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1),$$

$$S_{ABC} = \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1|$$

[Hint: use the previous part with $O = (0, 0)$.]



The picture on the right helps visualize the products that appear in the formula and their signs. Because of this picture, this formula is sometimes called “the shoelace formula”.

- * (c) Can you suggest an analog of this formula for a quadrilateral? for an n -gon?
- (d) Find the area of the quadrilateral with vertices at $(1, 3)$, $(1, 1)$, $(2, 1)$, and $(2021, 2022)$.