MATH 10
ASSIGNMENT 6: ANGLES BETWEEN LINES AND PLANES
OCT 27, 2024

Recal from the last time: dot product of two vectors is defined by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \bullet \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = x_1x_2 + y_1y_2 + z_1z_2$$

The dot product is symmetric ($\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$), linear as function of $\mathbf{v}$, $\mathbf{w}$, and satisfies $\mathbf{v} \bullet \mathbf{v} = |\mathbf{v}|^2$, or, equivalently, $|\mathbf{v}| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$. Moreover,

$$\mathbf{v} \bullet \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cos \varphi$$

where φ is the angle between vectors $\mathbf{v}$, $\mathbf{w}$ (in particular, $\mathbf{v} \bullet \mathbf{w} = 0$ if and only if $\mathbf{v} \perp \mathbf{w}$).

The last property is commonly used to find the angle between two vectors:

$$\cos \varphi = \frac{\mathbf{v} \bullet \mathbf{w}}{|\mathbf{v}| \cdot |\mathbf{w}|} \tag{1}$$

Equation of a line

Let us consider lines in the cordinate plane.

Theorem 1. *The equation of the line which is perpendicular to vector* $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ *and goes through point* $P = (x_0, y_0)$ *is*

$$a(x - x_0) + b(y - y_0) = 0$$

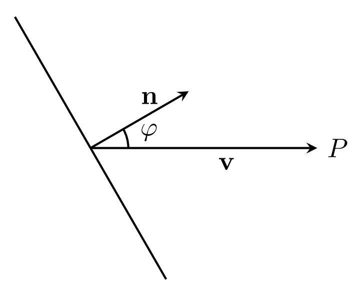
Conversely, if a line is given by equation $ax + by = d$*, then it is perpendicular to vector* $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$.

It gives us a way to compute the **angle between two lines**: it is equal to the angle between the perpendicular vectors to these lines, which can be computed using dot product.

It also gives a way to compute a **distance from a point** $P = \begin{bmatrix} x \\ y \end{bmatrix}$ **to a line**: if $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is an arbitrary point on the line, then the distance is equal to the length of the projection of vector $\mathbf{v} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$, on the perpendicular to the line

$$\text{distance} = |\mathbf{v}||\cos \varphi| = \frac{|\mathbf{v} \bullet \mathbf{n}|}{|\mathbf{n}|}$$

where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line.



Equation of the plane

The results above can be repeated, with very little changes, to planes in 3d:

Theorem. *The equation of the plane which is perpendicular to vector* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *and goes through point* $P = (x_0, y_0, z_0)$ *is a*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Conversely, if a plane is given by equation $ax + by + cz = d$*, then it is perpendicular to vector* $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

It gives us a way to compute the angle between two planes: it is equal to the angle between the perpendicular vectors to these planes, which can be computed using dot product.

It also gives a way to compute a distance from a point to a plane (see problem 5 below).

Homework

In all the problems where you are asked to find an angle, it is enough to find the cosine or sine of that angle (and then think roughly how much the angle is).

1. Compute the angle between the lines $2x + y = 2$ and $x + 2y = 0$.

2. Write the equation of the plane perpendicular to vector $\mathbf{v} = (1, 2, 1)$ and passing through the point $(1, 0, 0)$.

3. Find the angle between planes $2x - y - 3z + 5 = 0$ and $x + y - 2z = 3$.

4. Find the angle between the plane $x + y + z = 1$ and the x-axis.

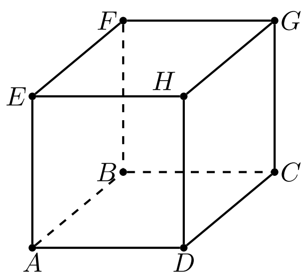
5. (a) Prove that the distance from point $A = (x_1, y_1)$ to the line given by equation $ax + by = d$ is

$$\text{distance} = \frac{|ax_1 + by_1 - d|}{\sqrt{a^2 + b^2}}$$

(b) Write and prove similar result for the plane in 3d.

6. Find the distance from the origin to the plane $x + y + z = 1$.

7. (Optional Problem) Consider the cube $ABCDEFGH$ (see figure)



(a) If we introduce a coordinate system such that A is the origin, and edges of the cube go along the coordinate axes, what is the equation of plane BED?
[Hint: it must be of the form $ax + by + cz = d$]

(b) Prove that this plane is perpendicular to the diagonal AG

(c) Find the distances between this plane and points A, G

(d) Find the angle between this plane and face $ABCD$

(e) Prove that the plane FHC is parallel to EBD. Find the distance between the two planes.

(f) Find the angles that the diagonal of the cube AG makes with the face diagonals AC and BD.