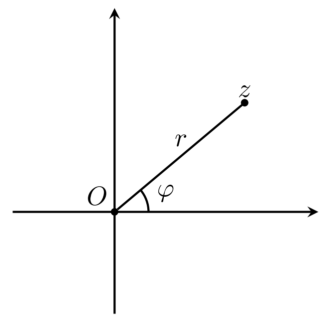
MATH 10
ASSIGNMENT 3: COMPLEX NUMBERS: DE MOIVRE FORMULA
SEPTEMBER 29, 2024

Magnitude and argument of a complex number



The magnitude of a complex numbers $z = a + bi$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$; geometrically it is the length of vector $z = (a, b)$. If $z \neq 0$, its *argument* $\arg z$ is defined to be the angle between the positive part of x-axis and the vector z measured counterclockwise. Thus, instead of describing a complex number by its coordinates $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$ we can describe it by its magnitude $r = |z|$ and argument $\varphi = \arg(z)$:

Relation between r, φ and $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$ is given by

$$a = r\cos(\varphi), \qquad b = r\sin(\varphi)$$
$$z = a + bi = r(\cos(\varphi) + i\sin(\varphi))$$

Thus, one can write the complex number with magnitude r and argument φ as

$$z = r(\cos\varphi + i\sin\varphi).$$

Geometric meaning of multiplication

Theorem.

1. *If z is a complex number with magnitude 1 and argument φ, then multiplication by z is rotation by angle φ:*

$$z \cdot w = R_\varphi(w)$$

where R_φ is operation of **counterclockwise** *rotation by angle φ around the origin.*

2. *If z is a complex number with absolute value r and argument φ, then multiplication by z is rotation by angle φ and rescaling by factor r:*

$$z \cdot w = rR_\varphi(w)$$

Addition of argument

Theorem. *When we multiply two complex numbers, magnitudes multiply and arguments add:*

$$|z_1 z_2| = |z_1| \cdot |z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \mod 360^\circ$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad \arg(\tfrac{z_1}{z_2}) = \arg(z_1) - \arg(z_2) \mod 360^\circ$$

In particular, this implies that if $z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

This is known as De Moivre's formula.

Homework

1. Show that
   (a) $|\overline{z}| = |z|$, $\arg(\overline{z}) = -\arg(z)$
   (b) Show that $\frac{\overline{z}}{z}$ has magnitude one. What is its argument if argument of z is φ?
   (c) Check part (b) for $z = 1 + i$ by explicit calculation.

2. If z has magnitude 2 and argument $3\pi/2$, and w has magnitude 3 and argument $\pi/3$, what will be the magnitude and argument of zw? Can you write it in the form $a + bi$?

3. Which transformations of the complex plane are given by the formulas

(a) $z \to iz$	(b) $z \to (1 + i\sqrt{3})z$	(c) $z \to \dfrac{z}{1+i}$
(d) $z \to \dfrac{z + \overline{z}}{2}$	(e) $z \to (1 - 2i + z)$	(f) $z \to \dfrac{z}{\lvert z\rvert}$
(g) $z \to i\overline{z}$	(h) $z \to -\overline{z}$	

Draw the image of the square $0 \le \operatorname{Re} z \le 1$, $0 \le \operatorname{Im} z \le 1$ under each of these transformations.

4. Let $p(x)$ be a polynomial with real coefficients.
   (a) Show that for any **complex** z, we have $\overline{p(z)} = p(\overline{z})$.
   (b) Show that if z is a complex root of p, i.e. $p(z) = 0$, then $\overline{z}$ is also a root.
   (c) Show that if $p(z)$ has odd degree and completely factors over $\mathbb{C}$ (i.e. has as many roots as is its degree), then it must have at least one real root.

5. Consider the equation $x^3 - 4x^2 + 6x - 4 = 0$.
   (a) Solve this equation (hint: one of the roots is an integer).
   (b) Find the sum and product of the roots in two ways: by using Vieta formulas and by explicit computation. Check that the results match.

6. Using the argument addition rule, derive a formula for $\cos(\varphi_1 + \varphi_2)$, $\sin(\varphi_1 + \varphi_2)$ in terms of sin and cos of φ_1, φ_2. [Hint: let $z_1 = \cos\varphi_1 + i\sin\varphi_1$, $z_2 = \cos\varphi_2 + i\sin\varphi_2$; then $z_1 z_2 =$?]

7. Compute

$$(3+4i)^{-1}, \qquad (1-i)^{12}, \qquad (1-i)^{-12}, \qquad \left(\frac{1+i}{1-i}\right)^{2024}, \qquad (i\sqrt{3}-1)^{17}$$

8. Using de Moivre's formula, write a formula for $\cos(3\varphi), \sin(3\varphi)$ in terms of $\sin\varphi$, $\cos\varphi$.

9. (a) Write out following sets in English:
      (i) $\{z \mid z^5 = 1\}$
      (ii) $\{w \mid w\overline{w} = 2\}$
      (iii) $\{u \mid \exists k \in \mathbb{N}, |u| = 2k\}$
   (b) Write out the following sets using mathematical notation:
      (i) All numbers where their argument is a multiple of $\frac{\pi}{6}$.
      (ii) The numbers that are equal to their conjugate.

***10.** Compute $1 + \cos\varphi + \cos 2\varphi + \cdots + \cos n\varphi$. [Hint: if $z = \cos\varphi + i\sin\varphi$, what is $1 + z + z^2 + \cdots + z^n$?]