MATH 10
ASSIGNMENT 2: COMPLEX NUMBERS REVIEW
SEP 22, 2024

Complex numbers

Consider the set $\mathbb{R}[i]$ of polynomials with real coefficients in one variable (which we will now denote by i rather than x) but with one extra relation:

$$i^2 + 1 = 0.$$

Thus, we will treat two polynomials in i which differ by a multiple of i^2+1 as equal (This can be done more formally in the same was as we define multiplication and division of remainders modulo n for integers).

Note that this relation implies

$$i^2 = -1, \qquad i^3 = i^2 i = -i, \qquad i^4 = 1, \dots$$

so using this relation, any polynomial can be replaced by a polynomial of the form $a + bi$. For example,

$$(1+i)(2+3i) = 2 + 4i + 3i^2 = 2 + 4i - 3 = -1 + 4i.$$

Thus, we get the following definition:

Definition. The set $\mathbb{C}$ of complex numbers is the set of expressions of the form $a+bi$, $a, b \in \mathbb{R}$, with addition and multiplication same as for usual polynomials with an added relation $i^2 = -1$.

Since multiplication and addition of polynomials satisfies the usual distributivity and commutativity properties, the same holds for complex numbers.

Note that any real number a can also be considered as a complex number by writing it as $a + 0i$; thus, $\mathbb{R} \subset \mathbb{C}$.

It turns out that complex numbers can not only be multiplied and added but also divided (see problem 4).

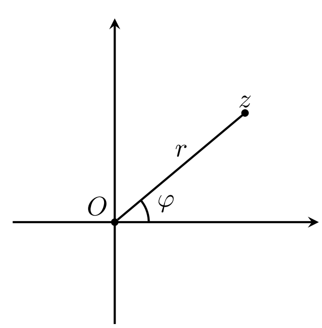
We can represent a complex number $z = a + bi$ by a point on the plane, with coordinates (a, b). Thus, we can identify

$$\text{complex numbers} = \text{pairs } (a,b) \text{ of real numbers} = \text{vectors in a plane}$$

In this language, many of the operations with complex numbers have a natural geometric meaning:

- Addition of complex numbers corresponds to addition of vectors.
- The magnitude (also called absolute value) $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$ is just the distance from the corresponding point to the origin, or the length of the corresponding vector. More generally, distance between two points z, w is $|z - w|$.
- Complex conjugation $z \mapsto \bar{z}$ is just the reflection around x-axis.

Magnitude and argument



The magnitude of a complex numbers $z = a + bi$ is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$; geometrically it is the length of vector $z = (a, b)$. If $z \neq 0$, its *argument* $\arg z$ is defined to be the angle between the positive part of x-axis and the vector z measured counterclockwise. Thus, instead of describing a complex number by its coordinates $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$ we can describe it by its magnitude $r = |z|$ and argument $\varphi = \arg(z)$:

Relation between r, φ and $a = \operatorname{Re}(z)$, $b = \operatorname{Im}(z)$ is given by

$$a = r\cos(\varphi), \qquad b = r\sin(\varphi)$$
$$z = a + bi = r(\cos(\varphi) + i\sin(\varphi))$$

Homework

1. Compute the following expressions involving complex numbers:

$$\begin{array}{ll}\text{(a) } (1+2i)(3+i) & \text{(b) } i^7\\ \text{(c) } (1+i)^2 & \text{(d) } (1+i)^7\\ \text{(e) } \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3 & \end{array}$$

2. Define for a complex number $z = a + bi$ its *conjugate* by $\bar{z} = a - bi$.

(a) Prove by explicit computation that $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} \cdot \bar{w}$.

(b) Prove that for $z = a + bi$, $z \cdot \bar{z} = a^2 + b^2$ and thus, it is non-negative real number.

3. Define for any complex number its absolute value by $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$ (see previous problem). Prove that then $|zw| = |z||w|$. [Hint: use formula $|z| = \sqrt{z\bar{z}}$ instead of $|z| = \sqrt{a^2 + b^2}$.]

4. Prove that any non-zero complex number z has an inverse: there exists w such that $zw = 1$ (hint: $z\bar{z} = |z|^2$).

5. Compute

$$\begin{array}{ll}\text{(a) } (1+i)^{-1} & \text{(b) } \dfrac{1+i}{1-i}\\ \text{(c) } (3+4i)^{-1} & \text{(d) } (1+i)^{-3}\end{array}$$

6. (a) Find a complex number z such that $z^2 = i$

(b) Find a complex number z such that $z^2 = -2 + 2i\sqrt{3}$.

[Hint: write z in the form $z = a + bi$ and then write and solve equation for a, b]

7. Find the absolute value and argument of the following numbers:

$1 + i$

$-i$

$w = \frac{\sqrt{3}}{2} + \frac{i}{2}$ (hint: show that the points 0, w, $\overline{w}$ form a regular triangle)

8. Find a complex number which has argument $\pi/4 = 45^\circ$ and absolute value 2.

9. Draw the following sets of points in $\mathbb{C}$:

(a) $\{z \mid \operatorname{Re} z = 1\}$

(b) $\{z \mid |z| = 1\}$

(c) $\{z \mid \arg z = 3\pi/4\}$ (if you are not familiar with measuring angles in radians, replace $3\pi/4$ by 135°).

(d) $\{z \mid \operatorname{Re}(z^2) = 0\}$

(e) $\{w \mid |w - 1| = 1\}$

(f) $\{w \mid |w^2| = 2\}$

(g) $\{z \mid z + \overline{z} = 0\}$

10. Find two numbers u, v such that

$$u + v = 6$$
$$uv = 13$$

Hint: use Vieta formulas.

11. For two numbers $z_1 = r_1(\cos(\varphi_1) + i\sin(\varphi_1))$ and $z_2 = r_2(\cos(\varphi_2) + i\sin(\varphi_2))$, find the formula for $z_1 z_2$.