Convex polytopes and their shadows

"Polytopes" are the generalization of polygons (squares, triangles, etc.) and polyhedra (cubes, octahedra, etc.) to any number of dimensions: they are geometric objects with "flat" sides. A four-dimensional polytope is also often called a "polychoron".

Some people have a difficult time visualizing higher-dimensional objects. In order to turn a higher dimensional object into something we can visualize, one approach is to draw its "shadow", also known as its "projection", on our lower-dimensional world. In order to make sense of this, we start by representing points inside a polytope by writing down their coordinates. For instance, a standard cube C can be thought of as the following collection of points:

$$C = \big\{(x, y, z) \text{ such that } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \text{ and } 0 \le z \le 1\big\}.$$

Its shadow/projection on the xy-plane is then defined to be

$$\mathrm{proj}_{xy}(C) \coloneqq \big\{(x, y) \text{ such that there exists a } z \text{ satisfying } (x, y, z) \in C\big\}.$$

From three dimensions to two

1. A standard octahedron O is defined to be the following collection of points:

   $$O = \big\{(x, y, z) \text{ such that } |x| + |y| + |z| \le 1\big\}.$$

   Compute the projection $\mathrm{proj}_{xy}(O)$ of the octahedron onto the xy-plane.

2. I've rotated a cube C_{rot} so that a point (x, y, z) is contained in C_{rot} if and only if it satisfies the system of inequalities

   $$\begin{aligned} -1 &\le 2x + 2y - z \le 1, \\ -1 &\le 2x - y + 2z \le 1, \\ -1 &\le -x + 2y + 2z \le 1. \end{aligned}$$

   (a) Compute the projection $\mathrm{proj}_{xy}(C_{\mathrm{rot}})$ onto the xy-plane.

   (b) For each point (x, y) in the shadow of C_{rot}, there is a least and greatest value of z such that $(x, y, z) \in C_{\mathrm{rot}}$. Compute the least such value of z as a function of (x, y) - what does this computation tell you about the shape of C_{rot}?

   (c) Is C_{rot} really a rotated cube?

3. I've chopped a standard cube C into several pieces based on how the coordinates (x, y, z) are sorted. One of the pieces, which I will call S, is defined by:

   $$S = \big\{(x, y, z) \text{ such that } 0 \le x \le y \le z \le 1\big\}.$$

(a) Compute the projections $\text{proj}_{xy}(S), \text{proj}_{xz}(S)$, and $\text{proj}_{yz}(S)$.

(b) Describe the shape of S.

(c) What is the volume of S?

4. I chopped off four of the corners of a standard cube C - specifically, I chopped off the corners where the sum of the coordinates $x + y + z$ is odd - and found myself left with the following shape T:

$$T = \{(x, y, z) \text{ such that } x + y + z \leq 2 \text{ and } x \leq y + z \text{ and } y \leq x + z \text{ and } z \leq x + y\}.$$

   (a) What is the projection $\text{proj}_{xy}(T)$ of T onto the xy-plane?

   (b) Find the smallest z such that $(x, y, z) \in T$, as a function of the point $(x, y) \in \text{proj}_{xy}(T)$.

   (c) Describe the shape of T, and find its volume.

Polychora

5. A standard 4-dimensional "orthoplex" O^4 is defined to be the following collection of points:

$$O^4 = \{(x, y, z, w) \text{ such that } |x| + |y| + |z| + |w| \leq 1\}.$$

   (a) Compute the projection $\text{proj}_{xyz}(O^4)$ of the orthoplex onto the three-dimensional xyz-space.

   (b) Find the smallest w such that $(x, y, z, w) \in O^4$, as a function of $(x, y, z) \in \text{proj}_{xyz}(O^4)$.

   (c) How many "cells"/"facets" (that is, three-dimensional analogues of faces) does the orthoplex O^4 have?

6. I've rotated a "tesseract" C^4_{rot} so that a point (x, y, z, w) is contained in C^4_{rot} if and only if it satisfies the system of inequalities

$$\begin{aligned} -1 &\leq x + y + z - w \leq 1, \\ -1 &\leq x + y - z + w \leq 1, \\ -1 &\leq x - y + z + w \leq 1, \\ -1 &\leq -x + y + z + w \leq 1. \end{aligned}$$

   Describe the projection $\text{proj}_{xyz}(C^4_{\text{rot}})$ onto three-dimensional xyz-space.

7. The "24-cell"/"octaplex" Q is the collection of points (x, y, z, w) satisfying the system of inequalities

$$\begin{aligned} |x| + |y| \leq 1, \quad & |x| + |z| \leq 1, \\ |x| + |w| \leq 1, \quad & |y| + |z| \leq 1, \\ |y| + |w| \leq 1, \quad & |z| + |w| \leq 1. \end{aligned}$$

   (a) Describe the projection $\text{proj}_{xyz}(Q)$ onto three-dimensional xyz-space.

   (b) How many vertices does Q have?

   (c) What is the largest possible value of $x^2 + y^2 + z^2 + w^2$ for points $(x, y, z, w) \in Q$?