

Introduction to Game Theory

Just as important as the ability to *solve* math problems is the ability to turn a real life situation into a math problem. This is called *mathematical modeling*. A good mathematical model simplifies the situation enough for you to be able to reason about it, but keeps enough of the important details for your conclusions about the model to be relevant to the real world.

Strategic planning and negotiation are some of the most complex situations we face on a daily basis. Game theory is the study of mathematical models for these situations. Every game we can study has many variations:

- we might be playing the game against an opponent we've never met before, and who we will never meet again,
- we might be playing the game against our classmate, who we will be interacting with in other ways for the rest of the year (and might hold a grudge),
- we might play the same game against our opponent over and over until the end of time,
- we might be allowed to discuss our plans with the other players before we actually play,
- we might be able to bribe the other players to let us win with promises of money or future favors,
- we might be able to sign legally binding contracts,
- we might be allowed to flip coins or roll dice to help us make random decisions,
- the people we are playing against might not be thinking very hard about their strategies, or might make mistakes.

Each variation leads to a different mathematical model, and the best strategy might change completely depending on which variation we study.

Classics

1. (The Prisoner's Dilemma) You and one of your associates have been captured by the police and are being interrogated separately. The police suspect that the two of you worked together to commit a major crime, but they have no evidence to prove it. You each have two options: confess, or stay silent.
 - If you both stay silent, then you both will be put in prison for one year (for a minor crime that they have evidence you both committed).

- If one of you confesses and the other stays silent, then the one who confesses goes free immediately, while the one who stayed silent goes to jail for ten years.
 - If you both confess, then you both go to jail for nine years.
- (a) Find a good way to organize the important details of the situation. What is your best strategy?
 - (b) A lot of people don't like the answer to part (a). Are there more realistic variations of this game that change the best strategy?
2. (Chicken) You and a rival driver are trying to prove your bravery by driving directly towards each other at high speed without swerving. You each only have a single instant to (simultaneously) make a choice between two options: swerve, or drive straight. Make a mathematical model for this situation, and attempt to analyze it.
 3. (Battle of the Sexes) In a world without cell phones, Alice and Bob would like to meet up somewhere to spend the evening together. There are two events going on tonight: a boxing match, and a ballet. Alice prefers boxing to ballet, while Bob prefers ballet to boxing, but neither one will be happy if they show up to either event alone.
 4. (Ultimatum Game) Two starving people on a deserted island come across a sealed twinkie at nearly the same time. The first person to see the twinkie makes an offer to the second person about how to split it up between the two of them. The second person then gets to decide whether to accept the offer, or to reject the offer by destroying the twinkie (so that neither person gets any of it).
 5. (Rock, Paper, Scissors) Two players simultaneously make hand signs for either a rock, a piece of paper, or scissors. Rock beats scissors, scissors beats paper, and paper beats rock (if they both make the same hand sign, it's a tie).
 - (a) Find a good way to organize the important details of the situation. What is the best strategy, and how often does it win?
 - (b) I have a friend who routinely wins large rock-paper-scissors tournaments (where each match is best out of three games). How is this possible?
 6. (Guess $2/3$ of the average) A large group of people each privately write down a number between 0 and 100 on a piece of paper, together with their name. We then collect all the pieces of paper, compute the average of all the numbers written, and give a prize to the person whose number is closest to $2/3$ of the average.
 7. (Dollar Auction) A dollar bill is being offered for sale at an auction with the following strange rule: both the winner of the auction and the second-highest bidder must pay the amount that they bid. At any time, any participant can increase their bid as long as they increase it to be at least one penny more than the currently highest bid. What can we expect to happen?
 8. (Hot Potato) Three people are playing a game involving a hot potato. One of the players starts off holding the potato. In each round of the game, whoever is currently holding the potato chooses one of the other two players to give the potato to. Whoever ends up holding the potato after the end of the third round explodes.

Less symmetrical games

9. In an exploration-themed board game, Alice and Bob both have to simultaneously choose compass directions: Alice must choose between North and South, while Bob must choose between West and East. Based on the combined direction they choose, they receive points as follows:
 - if they decide to go North-West, then Alice will get 0 points while Bob will get 3 points,
 - if they decide to go North-East, then Alice will get 3 points while Bob will get 0 points,
 - if they decide to go South-West, then Alice will get 2 points while Bob will get 1 points,
 - if they decide to go South-East, then Alice will get 1 points while Bob will get 2 points.
10. (Commitment) The setup is the same as before, but the moves are no longer simultaneous - now Bob must announce his choice before Alice - and the payoffs are different:
 - if they decide to go North-West, then Alice will get 3 points while Bob will get 3 points,
 - if they decide to go North-East, then Alice will get 2 points while Bob will get 4 points,
 - if they decide to go South-West, then Alice will get 4 points while Bob will get 1 points,
 - if they decide to go South-East, then Alice will get 1 points while Bob will get 2 points.

However, there is an additional twist: before Bob makes his choice, Alice can choose to publicly sign a binding contract to give some of her points to Eve in certain circumstances. Is there any way that this could somehow help Alice?

Utility

11. You have a classmate who strongly prefers apple pie to cherry pie, strongly prefers cherry pie to pumpkin pie, and strongly prefers pumpkin pie to apple pie. Can you take advantage of this situation?
12. (The Problem of Red and Black) Your father suddenly needs an expensive surgery, which costs \$1,000, in order to save his life from a medical condition that will kill him by tomorrow morning. You only have \$100 to your name, and no one is willing to give you a loan on such short notice. However, there is a casino down the street with a tiny roulette wheel with slots numbered from 1 to 6, with 1 and 4 colored green, 2 and 5 colored red, and 3 and 6 colored black. In each round of roulette, you can put any amount of the money you have betting on either of the colors red or black, doubling the money you put down if you correctly predict the color of the slot the roulette ball ends up in, and losing the money otherwise. You can play as many rounds as you like, as long as you don't run out of money. How should you play?
13. (Kelly Betting) Another casino opens up down the street with the same type of roulette wheel, but they stupidly changed the rules to the roulette game so that green slots are treated as if they were black slots. They aren't allowed to change the rules back until they let you play at least 100 rounds. Starting from \$100 dollars, how rich can you expect to get in this situation?
14. (St. Petersburg Paradox) A very rich person makes a strange offer to you: if you pay them $\$x$ now, they will repeatedly flip a coin until they get tails, and then pay you $\$2^n$, where n is the number of times they flipped heads. For which values of x would you accept their offer?

Difficult Games

15. (Colonel Blotto) Two generals are fighting a war which has three battle fronts. Each general has 100 soldiers to split between the three battle fronts, and neither has any spies to tell them about the other general's plans. At each battle front, whichever side has more soldiers will win the battle (the other side surrenders without a fight). The general who wins two battle fronts wins the whole war.
16. Two rich rivals are trying to one-up each other by buying expensive paintings at a sealed-bid auction. They both start out with a budget of \$1,000,000, and there are five paintings up for auction. In each round, they both privately write down a bid for the current painting, and whoever bids higher pays their bid and gets the painting. Whoever collects more paintings wins the competition.
17. Make up a mathematical model of a simplified version of poker, and try to analyze it.