

MATH CLUB: WIZARDS, ISLANDERS, AND COLLABORATIVE ALGORITHMS

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1. Black and White Stickers:

Ten students are in a classroom, wanting to leave, and the teacher decides to dismiss students according to their success in the following game: the teacher will put a black or a white sticker on each student's forehead (one per student, the color randomly chosen), so that the students can see each others' stickers but not their own. The teacher will then ask the students, one by one, to guess the color of their sticker. If they get it right, they may leave; if not, they must stay for the next class (which is very boring). During the game, starting from when the teacher places the first sticker, the students may not communicate at all, except that they may hear each others' guesses. The students are given five minutes to talk to each other before the game begins. They want to devise a strategy to save as many of them as possible. What's the best strategy? (Note: if anyone cheats, all students must stay for the next class.)

2. 48 mathematicians are captured by a warrior tribe. The king of the tribe decides to give them a test. He says: "I have 49 hats, seven of each seven colors: red, orange, yellow, green, blue, purple, and black. I will hide one hat and put the others on you, so that each of you can see all the hats but your own. After that, I will ask each of you in turn what color is his own hat. If at least 47 of you give me the correct answer, you all will be released. Otherwise, you will all be sacrificed to the gods."

As in the previous problem, the mathematicians are allowed to talk before the test to come up with a common strategy, but once the test begins, they must keep silence - except for answering the question about their own hat color. Each can hear responses of others.

Can the mathematicians avoid being sacrificed?

3. Wizards and hats

Three Masters of Logic wanted to find out who was the wisest amongst them. So they turned to their Grand Master, asking to resolve their dispute. "Easy," the old sage said. "I will blindfold you and paint either red, or blue dot on each man's forehead. When I take your blindfolds off, if you see at least one red dot, raise your hand. The one who guesses the color of the dot on his forehead first, wins." And so it was said, and so it was done. The Grand Master blindfolded the three contestants and painted red dots on every one. When he took their blindfolds off, all three men raised their hands as the rules required, and sat in silence pondering. After a short pause, one of them said: "I have a red dot on my forehead." How did he guess?

4. The Blue-Eyed Islanders

There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.

One evening, he addresses the entire tribe to thank them for their hospitality.

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What effect, if anything, does this have on the tribe?

[Hint: consider first what would happen if there was only one blue-eyed islander; then, only two.]