MATH CLUB: FIBONACCI NUMBERS AND OTHER RECURRENT SEQUENCES

NOV 17, 2024

In many problems, a sequence is defined using a recurrence relation, i.e. the next term is defined using the previous terms. By far the most famous of these is the Fibonacci sequence:

$$F_0 = 0, F_1 = F_2 = 1, \qquad F_{n+1} = F_n + F_{n-1} \tag{1}$$

The first several terms of this sequence are below:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34, \dots$$

For such sequences there is a method of finding a general formula for nth term, outlined in problem 7 below.

1. Into how many regions do n lines divide the plane? It is assumed that no two lines are parallel, and no three lines intersect at a point.
Hint: denote this number by R_n and try to get a recurrent formula for R_n: what is the relation betwen R_n and R_{n-1}?

2. How many ways are there to cut a 2×20 strip of paper into 1×2 "dominos"?
Hint: again, write a recurrence formula!

3. How many ways are there to write a 10-letter "word" consisting of letters A and B if we do not allow letter B to appear two times in a row? What if we allow for B at most two times in a row?

4. Prove that the Fibonacci numbers satisfy the following identity: $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$
Can you guess a formula for the sum $F_1^2 + F_2^2 + \dots + F_n^2$?

5. This problem is for those who are familiar with matrix multiplication.
Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

so that $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix}$

Compute A^2, A^3, A^4 and try to guess a general formula for A^n

6. (a) Prove the following formula

$$F_{n+m} = F_n F_{m+1} + F_{n-1} F_m$$

[This can be done in many ways. If you have done the previous problem, then the easiest way is to notice $A^{m+n} = A^m A^n$. Otherwise, you can just prove by induction.]

(b) Let $L_n = F_{2n}/F_n$. Prove that then, $L_n = F_{n+1} + F_{n-1}$. (The sequence L_n has its own name: they are called Lucas numbers.)

(c) (AMC 12B, 2024).
Compute

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \dots + \frac{F_{20}}{F_{10}}$$

7. In this problem, we show how one can derive a formula for Fibonacci number.
Let us call a sequence a_n a *generalized Fibonacci sequence* (GFC) if it satisfies the same recurrence relation ($a_{n+1} = a_n + a_{n-1}$), but might have different first two terms.

(a) Show that a geometric progression $a_n = \lambda^n$, $\lambda \neq 0$, is a GFC if and only if λ satisfies the equation

$$\lambda^2 = \lambda + 1 \tag{2}$$

Find the roots of this equation.

(b) Let λ_1, λ_2 be the two roots of equation (2). Show that then any sequence of the form

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n \tag{3}$$

(where c_1, c_2 are some constants that do not depend on n) is a GFC.

(c) Find constants c_1, c_2 so that the sequence a_n defined by (3) satisfies $a_0 = 0$, $a_1 = 1$.
(d) Write a general formula for F_n.

8. Use a calculator to estimate how large F_{1000} is.

9. Pell numbers are defined by the relations

$$P_0 = 0, P_1 = 1, \qquad P_{n+1} = 2P_n + P_{n-1} \tag{4}$$

Compute several Pell's numbers by hand; then try to modify the method of problem 7 to get a formula for Pell's numbers.

10. Show that for large n, the ratio $(P_{n-1} + P_n)/P_n$ is close to $\sqrt{2}$. Write the approximation one gets in this way for $n = 8$ and check how close it is to the actual value. (This series of approximations to $\sqrt{2}$ was known already in 4th century BC).