MATH CLUB: SEMI-INVARIANTS

NOV 3, 2024

In many problems, the problem is greatly simplified if you notice that there is some quantity which always changes in the same direction (e.g., always increases).

1. We are given a 100×100 table, in each square of which we have an integer number. You can flip the signs of all numbers in one row or in one column of this table.

Prove that using this operation, in finitely many steps you can make sum of each row and each column non-negative.

2. We are given n red dots and n black dots on the plane, no three of them on the same line. Prove that then, one can connect them by n non-intersecting segments, each connecting a red dot with a black one.

3. 101 astronomers are standing in a circle and arguing. Some of them believe that Pluto is a planet, and others, that it is not. Once a minute they all announce their opinions. Immediately after that, each astronomer both of whose neighbors disagree with him, changes his opinion, convinced by his neighbors. Prove that this can't continue indefinitely: after some time, no more changes of opinion will happen. [Hint: two astronomers of the same opinion standing next to each other will never change their opinions.]

4. We have 4 pegs placed on the plane at corners of a square with side 1. At any time, you can take one of the pegs and "jump" over another, placing the first peg on opposite side of hte second, at same distance. (The second peg doesn't move). Is it possible to move all 4 pegs so that they are now at vertices of a square with side 2? with side 3?

5. A hotel has inifnite number of rooms, indexed by integers (including negative integers). At some moment, a group of 100 students move into the hotel, taking rooms at random (it is possible that some students share a room) However, each student plays loud music, and their music tastes are all different. As a result, they start changing rooms: if there are two students in rooms k, $k+1$, they may decide to move further apart, moving to rooms $k-1$, $k+2$ respectively. (Students living in the same room do not bother each other)

Prove that this can't continue indefinitely: after some number of moves, no more moves will be possible. What is the maximal number of moves possible?

***6.** 10 kids are sitting at a round table and dividing candy. At a signal, each of them gives some of his candy to his neighbor to the right: if he had even number of pices, he gives half of them; if he has $2k+1$ pieces, he gives $k+1$.

Prove that if the total number of candy is 100 pieces, then after some time each of them will have 10 pieces.

7. A $100{\times}100$ yard field of wheat is divided into 1 yd $\times$ 1 yd squares. Initially, 99 of these squares were infected by some crop disease. The disease spreads as follows: for every square, if in the given year at least 2 of its 4 neighbors were infected, then next year the infection spreads to this square. (The squares that were infected stay infected forever). Prove that the disease will never spread to the whole field.

Hints to all problems are available — just ask!