Homework 16

Lorentz transformations.

The result of Michelson-Morley experiment became clear after the work published by A. Einstein in 1905. In this work he demonstrated the constancy of the speed of light in all inertial reference frames.

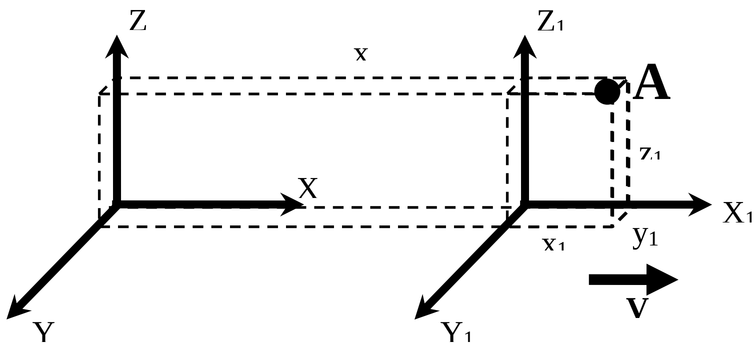

Fig.1

Let us consider a coordinate system $X_1Y_1Z_1$ which is moving at a velocity ***V*** with respect to the XYZ frame along the x axis (Fig.1). Let us imagine that the object A has coordinates x,y,z in the XYZ reference frame. We are going to express the coordinates x_1, y_1, z_1 of A in $X_1Y_1Z_1$ reference frame. According to classical nonrelativistic mechanics the transformation rules are:

$$\begin{aligned} x_1 &= x - Vt \\ y_1 &= y \\ z_1 &= z \\ t_1 &= t \end{aligned} \qquad (1)$$

This coordinal transformations are called Galilean transformations. The last equation looks absolutely trivial and intuitive. It means that the time flow is the same in both reference frames. If two events happen at the same time in one reference frame, they are expected to happen simultaneously in any other reference frame. A bit later, we will see that this is not true.

Michelson-Morley experiment suggested that the speed of light is the same in all inertial reference frames. This is in contradiction with the simple velocity composition rule which follows from the transformation rules (1) which are called Galilean transformations. Looks like the expressions (1) has to be corrected. Albert Einstein suggested the corrections. In a nice book "Basic concepts in relativity and early quantum theory" by R. Resnick and D. Halliday we read:

"In a paper entitled "Conversations with Albert Einstein" R. S. Shankland writes: "I asked Professor Einstein how long he had worked on the Special Theory of Relativity before 1905. He told me that he had started at age 16 and worked for ten years; first as a student when, of course, he could spend only part-time on it, but the problem was always with him. He abandoned many fruitless attempts, 'until al last it came to me that time was suspect'" What was it about time that Einstein questioned? It was the assumption, often made unconsciously and certainly not stressed that there exist a universal time that is the same for all observers<…> In prerelativistic discussions, the

assumptions was there implicitly by the absence of a transformation equation for t in the Galilean equations."

Albert Einstein
(1879-1955).

In class we discussed the way to obtain the correct expressions. First, we assumed that space and time are homogeneous. This means that all pints in space and moments of time are equivalent. It follows from the homogeneity assumption that the result of a measurement of a length or a time interval between two specific events should not depend on where or when the interval happens to be in our reference frame. Second, we used postulates of the special theory of relativity:

1. All the inertial reference frames are equivalent
2. The speed of light in vacuum is the same in all the reference frames.
   Then, the correct transformations are:

$$
\begin{aligned}
x_1 &= \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \\
y_1 &= y \\
z_1 &= z \\
t_1 &= \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}
\end{aligned}
\qquad (2)
$$

Here, c is the light speed in vacuum. These formulae – Lorentz transformations - were originally suggested by Hendrick Lorentz (1899) and, independently, by Joseph Larmor (1897). The most interesting feature in the formulae (2) is that now the time "flow" now depends on the velocity of the reference of frame.
As long as the velocity of the moving reference frame is equal to the speed of light the denominators of the first and last transformations in (2) become zero – this means that the Lorentz transformation do not make sense.

Lorentz transforms can be written in a more compact way if we introduce the following notations:

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}; \quad \beta = \frac{V}{c} \qquad (3).$$

γ is always more than 1, β is always less than 1.

Then, we have:

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(-\beta ct + x) \end{cases} \qquad (4)$$

Again, in expression (4) x' and t' are position and time as "seen" by an observer in the moving reference frame, while x and t are position and time for an observer at rest.

Let us consider how do the Lorentz transformation lead to the time dilation. Imagine the clock at rest at a certain point with a coordinate *x*. An observer in the clock reference frame measures a time interval $\Delta t=t_2-t_1$. We are moving with respect to the clock along the x-axis at a speed *V*. Also imagine, that the observer at rest sends us signals in the beginning and the end of Δt. We are measuring time interval $\Delta t'$ between the signals using a clock which are moving with us. How is our measured time interval $\Delta t'$ related to time interval Δt (the proper time interval)? We can use the first expression from (4), to express the beginning time t1 for the observer inn a moving frame through the time and position in a rest frame:

$$ct'_1 = \gamma(ct_1 - \beta x_1) \qquad (5).$$

Similar expression we can write for t2:

$$ct'_2 = \gamma(ct_2 - \beta x_2) \qquad (6).$$

Now we can subtract (5) from (6) and obtain:

$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x) \qquad (7)$$

But for the clock at rest $\Delta x=0$, so formula (7) becomes the familiar expression for the time dilation:

$$\Delta t' = \gamma \Delta t$$

Just to remind, that the observer moving along x with respect to the clock which is on the ground sees the clock moving backward, along -*x*. So the time interval measured by this observer will be longer that the same interval measured by the observer in the clock reference frame, where the clock is at rest.

Now, let us consider length contractions. Imagine that we have a thin long rod of length $l=x_2-x_1$, oriented along x axis. x_2 and x_1 are the coordinates of the rod's ends. The rod is at rest with respect to the ground. We are moving with respect to the rod with a speed *V* along the *x*-axis. What is the length of the rod in our reference frame?

We see the rod moving backward, along -*x* direction. Again, we can use expressions (4). But now we have to set $\Delta t'=0$, since the rod is moving with respect to us, and, to measure the rod length accurately, we have to measure the coordinate of the rod ends in the same moment of time:

$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x) = 0 \implies c\Delta t - \beta \Delta x = 0 \qquad (8).$$

so $\Delta t = \frac{\beta}{c}\Delta x$ (9)

Then we can write the second equation from (4):

$$\Delta x' = \gamma(-\beta c\Delta t + \Delta x) \qquad (10)$$

We substitute Δt from equation (9):

$$\Delta x' = \gamma\left(-\beta c\frac{\beta}{c}\Delta x + \Delta x\right) = \gamma(-\beta^2\Delta x + \Delta x) = \gamma(1-\beta^2)\Delta x \qquad (11).$$

But

$$\gamma(1-\beta^2) = \frac{1}{\gamma} \qquad (12).$$

So, we obtain the familiar expression for the length contraction:

$$\Delta x' = \frac{\Delta x}{\gamma} \qquad (13), \text{ or}$$

$$l' = \frac{l}{\gamma} \qquad (14).$$

Here, *l'* is the length of the rod (as we "see" it) which is mowing with respect to us, *l* is he length of the rod at rest. Length contraction "works" only in the direction of motion. The lengths along the directions, perpendicular to the direction of motion remain intact.

Problems:

In the problems below ***c*** is the speed of light in vacuum.

1. Imagine that an astronaut is moving at a speed 0.8c relative to the Earth. Find how long is his (her) hour as it is "seen" from the Earth?

2. How do we "see" the volume of a cube with the side of 1m at rest if the cube is moving with respect to us along one of its edges at a speed of 0.5c?

3. Two events happen at the same time in a certain reference frame. Do you think that these events are simultaneous in all inertial reference frames? Proof your answer.

4. In a certain reference frame two events happen at different moments of time in different places. Is it always possible to find another inertial reference frame in which these events will happen at the same place?