Homework 18

Relativistic inertia.

To start we consider a simple absolutely inelastic collision of two identical balls with a mass ***m*** mowing toward each other with identical speeds ***u*** (with respect to an observer1, Figure1). This approach is given in a nice book “Basic concepts in relativity and early quantum theory “ by Robert Resnick and David Halliday.

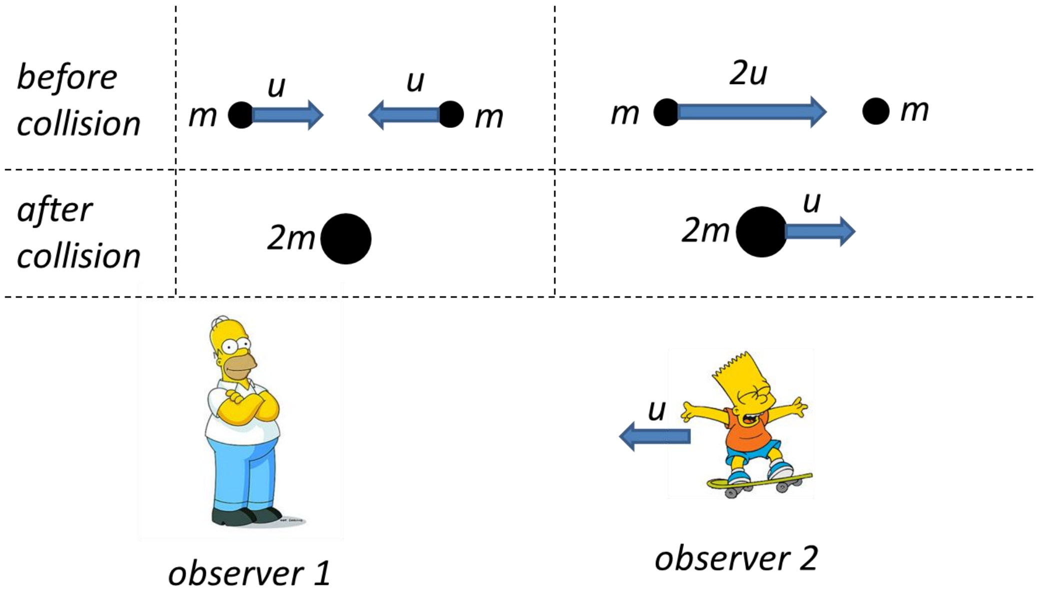

Figure 1.

After the collision the balls stick together and stop according to the momentum conservation law, stating that total momentum before collision (which is 0) has to be equal to the total momentum after collision. Observer 2, moving at a velocity u with respect to the observer 1 (see Figure 1) will see the ball moving at a velocity ***2u*** hitting another ball which is at rest. The total momentum of the balls before collision is ***2mu***, so after collision the big ball will be moving at a velocity ***u***. Everything looks fine at a first glance. And it *is* fine as long as observer 2 moves at a speed which is much less than the speed of light in vacuum ***c***. If this is not the case and ***u*** is comparable with ***c***, we cannot use the classic velocity composition rule and the speed of the left ball will not be measured as ***2u*** by the observer 2. Instead it will be :

$$u' = \frac{2u}{1+\frac{u^2}{c^2}} \qquad (1)$$

We can easily see that in this case the momentum conservation law seems to be broken for the observer 2. Instead of assuming that the momentum conservation law does not work, let us assume that the expression for classical momentum has to be changed for rapidly moving objects. As long as we know that the length and time intervals measured by observers moving with respect to each other are different (I mean time dilation and length contraction), it seems possible that the measured mass of the object depends on the velocity of the "measurer" with respect to the object. Let us find this dependence.

Let us denote as $\boldsymbol{m_0}$ the inertia of the particle in a reference frame where this particle is at rest; $\boldsymbol{m}$ will be the inertia of a particle moving at a speed $\boldsymbol{u}$, and $\boldsymbol{m'}$ will be the inertia of the particle moving at a speed $\boldsymbol{u'}$. (I used the word "speed" because we believe that due to symmetry considerations the mass of the particle does not depend on the direction of the velocity, and the word "inertia" instead of "mass" – for the reasons which I will explain in the end of the text). The inertia depending on the speed we will call *relativistic* inartia. So, Figure 1 will be slightly changed.

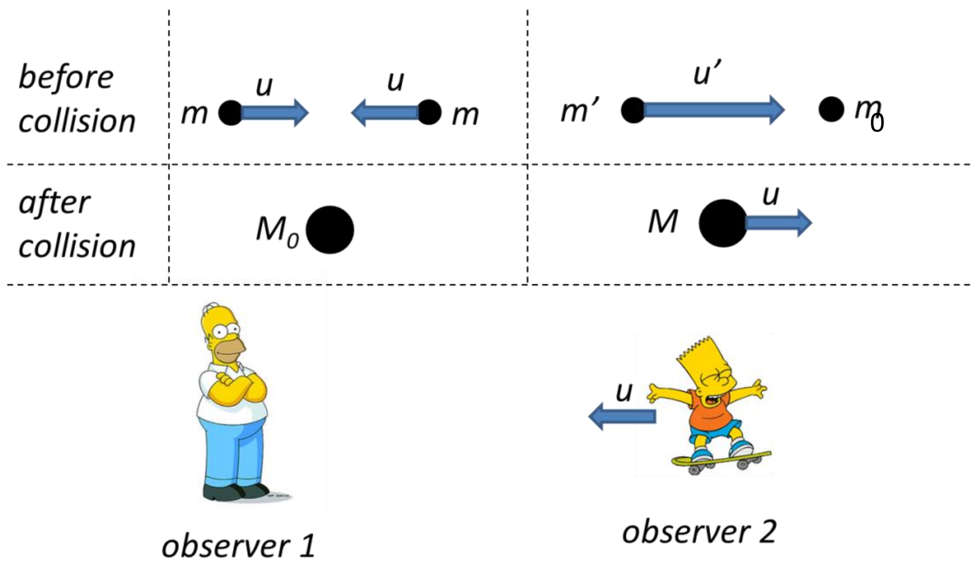

Figure 2.

Now we assume that correct formula for the momentum stays the same – the product of inertia and velocity, but, instead of constant inertia we will use the relativistic inertia which depends on the speed of the object. This new momentum we, again, will call "relativistic". It is the relativistic momentum which conserves. For observer 2 the relativistic momentum conservation gives:

$$Mu = m'u' \qquad (2)$$

Next, we assume that relativistic inertia conserves. Observer 2 will see that

$$M = m' + m_0 \qquad (3)$$

Now we have 3 equations (1,2 and 3 ☺) and five variables: $\boldsymbol{m_o}$, $\boldsymbol{m'}$, $\boldsymbol{M}$, $\boldsymbol{u}$, $\boldsymbol{u'}$. To understand how does the relativistic inertia depend on speed, we have to exclude $\boldsymbol{M}$ and $\boldsymbol{u}$ from the equations and obtain expression containing just $\boldsymbol{m_o}$, $\boldsymbol{m'}$, and $\boldsymbol{u'}$. From (2) and (3) we obtain:

$$u = u'\left(\frac{m'}{m'+m_0}\right) \qquad (4)$$

After substitution u in the expression (1) using formula (4) we will obtain:

$$u' = \frac{\frac{2u'm'}{m'+m_0}}{1+\frac{\left(\frac{u'm'}{m'+m_0}\right)^2}{c^2}} \implies 1 = \frac{\frac{2m'}{m'+m_0}}{1+\frac{\left(\frac{u'm'}{m'+m_0}\right)^2}{c^2}} \implies 1 + \left(\frac{m'}{m'+m_0}\right)^2 \cdot \frac{u'^2}{c^2} = \frac{2m'}{m'+m_0} \implies$$

$$\left(\frac{m'}{m'+m_0}\right)^2 \cdot \frac{u'^2}{c^2} = \frac{m'-m_0}{m'+m_0} \implies \frac{u'^2}{c^2} = \frac{(m'-m_0)\cdot(m'+m_0)}{m'^2} = \frac{m'^2-m_0^2}{m'^2} = 1 - \frac{m_0^2}{m'^2}$$

And, finally:

$$m' = \frac{m_0}{\sqrt{1-\frac{(u')^2}{c^2}}} \qquad (5)$$

So as long as an object having inertia $\boldsymbol{m_0}$ in a rest reference frame is moving with respect to the observer with a speed $\boldsymbol{v}$, the inertia of the object measured by the observer (relativistic inertia) will be:

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (6)$$

$\boldsymbol{m_0}$ is called *rest mass*. As long as $v \ll c$, then $m \approx m_0$ and, again, we can use expression of classical mechanics. As v approaching $\boldsymbol{c}$, the relativistic inertia increases infinitely so it is impossible to accelerate an object with nonzero rest mass to the speed of light in vacuum.

Important! Many scientists believe that the concept of relativistic mass is, somehow confusing: mass m in formula (6) is not the mass you will use to calculate gravity force. Moreover, it is not relativistically invariant. That is why I use the word "inertia" instead of "mass".

Problems

1. Find the speed at which the relativistic inertia of the object is 2 times more than its rest mass.
2. You have 18g of water at temperature slightly higher than 0^0C. How does this mass change if you will heat it up to 100^0C? Give a rough estimate (take the kinetic energy of the water molecules as ~kT, where k is the Boltzmann constant, T is absolute temperature).