Homework for December 17, 2023.

Algebra.

Review the classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.
2. Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?
3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
    a. no letter will be put into the envelope with its correct address?
    b. only 1 letter will be put into the envelope with its correct address?
    c. only 2 letters will be put into the envelope with its correct address?
    d. only 3 letters will be put into the envelope with its correct address?
    e. only 4 letters will be put into the envelope with its correct address?
    f. all 5 letters will be put into the envelope with its correct address?
5. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
6. In a survey on the students' chewing gum preferences, it was found that
    a. 20 like juicy fruit.
    b. 25 like spearmint.
    c. 33 like watermelon.
    d. 12 like spearmint and juicy fruit.
    e. 16 like juicy fruit and watermelon.
    f. 20 like spearmint and watermelon.

g. 5 like all three flavors.
h. 4 like none.

How many students were surveyed?

Geometry.

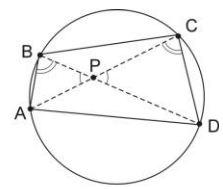
Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proofs of Ptolemy’s and Euclid’s theorems. Try solving the following problems including the unsolved problems from previous homeworks (problems marked with asterisk are optional – you are not expected to solve them).

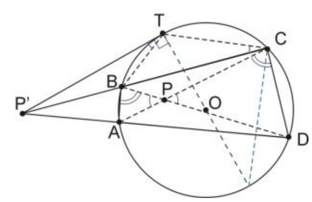
Problems.

1. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

   $|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2$,

   where $|PT|$ is a segment tangent to the circle (see Figure).



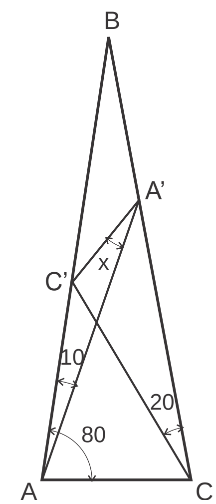


2. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?
3. Using the Ptolemy’s theorem, prove the following:
   a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.

b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.

c. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle?

4. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
5. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.
6. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



8. * Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).