MATH CLUB: INVARIANTS CONTINUED

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An invariant is something that does not change.

A semi-invariant is something that only changes in one direction (e.e.g, only decreases).

One of many applications of this idea is to the problems where various pieces are moved around on a board. In these cases, one can try to assign "weights" – non-negative real numbers – to each cell so that total sum of all weights occupied by the pieces stays constant during all moves.

You might also find the formula for the sum of (infinite) geometric progression useful:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad \text{for } |r| < 1.$$

1. We have an infinite sheet of square ruled paper (think of it as first quadrant on the coordinate plane), with cells indexed by pairs of positive integers. In the beginning, we have a chip on square $(1, 1)$. At every moment, we can make the following move: if there is a chip at square (i, j), and squares above and to the right of it (that is, squares $(i + 1, j)$ and $(i, j + 1)$) are both empty, we can remove the chip from (i, j) and put a chip in each of the squares $(i, j + 1)$ and $(i + 1, j)$.

Using these moves, can we clear the 3×3 square in the corner?

2. Conway's soldiers — one-dimensional version. This game is a variant of peg solitaire. You have an infinite line consisting of squares indexed by integers. Some cells are occupied by game pieces ("soldiers"). As in peg solitaire, a move consists of one soldier jumping over an adjacent soldier into an empty cell, and removing the soldier which was jumped over.

Initially all cells with index $i \geq 0$ are empty, and all cells with negative indices are filled with soldiers. The goal of the game is to place a soldier as far to the right as possible.

What is the highest position one can reach?

[Hint: try assigning weights which form a geometric progression.]

3. Conway's soldiers — two-dimensional version

This is similar to above, but now the soldiers are placed on two-dimensional infinite checkerboard. The board is divided by a horizontal line that extends indefinitely. Above the line are empty cells and all cells below the line are filled by "soldiers". As before, a move consists of one soldier jumping over an adjacent soldier into an empty cell, vertically or horizontally (but not diagonally), and removing the soldier which was jumped over. The goal of the game is to place a soldier as far above the horizontal line as possible.

(a) Give a examples of how one can reach rows 1, 2, 3.
(b) Can you reach row 4?
*(c) Prove that it is impossible reach row 5.