

MATH CLUB: COMBINATORICS 1

OCT 15, 2023

BASIC RULES

The number of ways to choose k items out of n if the order in which they are chosen matters is

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}.$$

The number of ways to choose k items out of n if the order in which they are chosen doesn't matter is

$${}_nC_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

For example, the number of sequences of length n consisting of k zeros and $n-k$ ones is ${}_nC_k$ (this is equivalent to choosing k positions where we put zeros).

1. How many ways there are to divide 10 students into two teams of 5? (Teams do not have names)
2. A language of some ancient tribe consists of 6 vowels and 8 consonants, and in each word, vowels and consonants must alternate. How many 9-letter words can there be in this language?
3. How many ways there are for 15 students to take seats in a classroom with 30 chairs?
4. How many ways are there to place 8 rooks on the chessboard so that no two can attack each other?
5. You are given m white balls and n black balls. How many way are there to arrange them in a line?
6. You are given m white balls and n black balls ($m > n$). How many way are there to arrange them in a line so that no two black balls are next to each other? [Hint: each black ball, except the last one, must be followed by a white one.]
- *7. You are given an unlimited supply of black and white balls. How many ways are there to create a line of 20 balls so that no two black balls are next to each other? What if you are allowed at most two black balls next to each other?

OVERCOUNTING

A simple idea is that if we want to count number of items in some set (e.g., number of ways to...) and we have found a method of counting that counts every item m times, then correct answer is (result of counting)/ m .

This is very commonly used in many problems.

10. How many different "words" can be formed by permuting letters of the word "Mississippi"? (by "word", we mean any sequence of letters, not necessarily meaningful).
11. How many ways there are to group $2n$ people into n pairs?
12. Lynne is making tokens for some game. Each token is a small cube, with faces colored in 6 colors (red, green, blue, yellow, black, white), one face of each color. How many different tokens are possible?
Same question for a token which has the shape of regular dodecahedron (12 faces, each a regular pentagon), using 12 colors.
- *13. How many different necklaces can one make using 1 green, 5 red, and 7 blue beads? using 2 green, 5 red, and 7 blue beads?