# MATH 8: EUCLIDEAN GEOMETRY

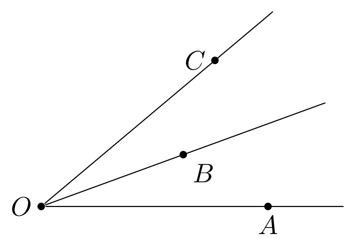
## 1. First axioms

After we introduced some objects, including undefined ones, we need to have statements (*axioms*) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the *minimal number* of such properties that we take for granted, just enough to be able to prove or derive harder and more complicated statements. Here are the first few axioms:

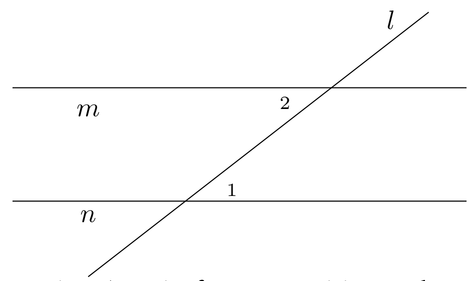
**Axiom 1.** *For any two distinct points $A, B$, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

**Axiom 2.** *If points $A, B, C$ are on the same line, and $B$ is between $A$ and $C$, then $AC = AB + BC$*

**Axiom 3.** *If point $B$ is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$.*




**Axiom 4.** *Let line $l$ intersect lines $m, n$ and angles $\angle 1$, $\angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*




In addition, we will assume that given a line $l$ and a point $A$ on it, for any positive real number $d$, there are exactly two points on $l$ at distance $d$ from $A$, on opposite sides of $A$, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

## 2. First theorems

Now we can proceed with proving some results based on the axioms above.

**Theorem 1.** *If distinct lines $l, m$ intersect, then they intersect at exactly one point.*

*Proof.* Proof by contradiction: Assume that they intersect at more than one point. Let $P, Q$ be two of the points where they intersect. Then both $l, m$ go through $P, Q$. This contradicts Axiom 1. Thus, our assumption (that $l, m$ intersect at more than one point) must be false. □

**Theorem 2.** *Given a line $l$ and point $P$ not on $l$, there exists a unique line $m$ through $P$ which is parallel to $l$.*

*Proof.* Here we have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

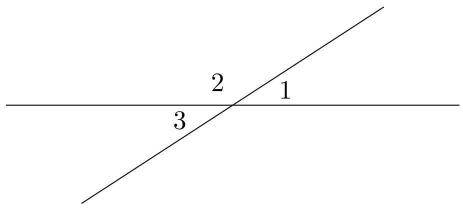
**Existence:** Let $m$ be any line that goes through $P$ and intersect $l$ at point $O$. Let $A$ be a point on the line $l$. Then we can measure the angle $\angle POA$. Now, let $PB$ be such that $m\angle BPO = m\angle POA$ and $B$ is on the other side of $m$ than $A$. In this case, by Axiom 4, $\overleftrightarrow{AB} \parallel l$.

**Uniqueness:** Imagine that there are two lines $m, n$ that are parallel to $l$ and go through $P$. Take a line $k$ that goes through $P$ and intersects $l$ in point $O$. Let $A$ be a point on line $l$ distinct from $O$, and $B, C$ — points on lines $m$ and $n$ respectively on the other side of line $k$ than $A$. Since both $m, n$ are parallel to $l$, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{CP}$ are the same — contradiction to our assumption that there are two such lines. □

**Theorem 3.** *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

*Proof.* Assume that $l$ and $n$ are not parallel and intersect at point $P$. But then it appears that there are two lines that are parallel to $m$ are go through point $P$ — contradiction with Theorem 2. □

**Theorem 4.** *Let $A$ be the intersection point of lines $l, m$, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*




*Proof.* Let angle $2$ be as shown in the figure to the left. Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

**Theorem 5.** *Let $l, m$ be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines $l, m$ are perpendicular and write $l \perp m$.)*

*Proof.* Left as a homework exercise. □

**Theorem 6.** *Let $l_1, l_2$ be perpendicular to $m$. Then $l_1 \parallel l_2$.*

*Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.*

*Proof.* Left as a homework exercise. □

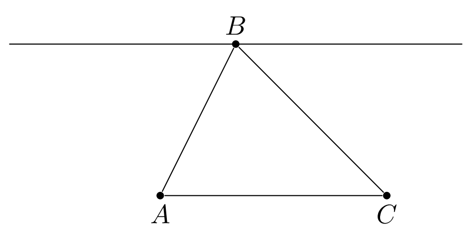
**Theorem 7.** *Given a line $l$ and a point $P$ not on $l$, there exists a unique line $m$ through $P$ which is perpendicular to $l$.*

*Proof.* Left as a homework exercise. □

## 3. Triangles

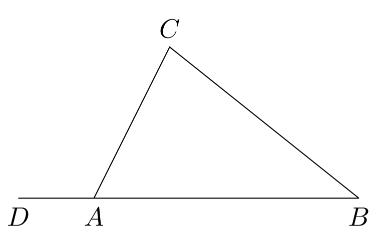
**Theorem 8.** *Given any three points $A$, $B$, $C$, which are not on the same line, and line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a triangle, written $\triangle ABC$. The three respective angles are called the triangle's interior angles.)*

*Proof.* The proof is based on the figure below and use of Alternate Interior Angles axiom. Details are left to you as a homework.




□

Given a triangle $\triangle ABC$, let $D$ be a point on the line $AB$, so that $A$ is between $D$ and $B$. In this situation, angle $\angle DAC$ is called an external angle of $\triangle ABC$.




**Theorem 9** (External angle theorem)**.** $m\angle DAC = m\angle B + m\angle C$ *(in particular this implies that $m\angle DAC > m\angle B$, and similarly for $\angle C$).*

*Proof.* $m\angle DAC = 180° - m\angle CAD$. But $m\angle CAD = 180° - m\angle C - m\angle B$, and therefore $m\angle DAC = m\angle B + m\angle C$. As a consequence, we see that $m\angle DAC > m\angle B$ and $m\angle DAC > m\angle C$. □

## 4. Congruence

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points $A$, $B$ is the same as the distance between points $C$, $D$, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \implies \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

## 5. Congruence of Triangles

Triangles consist of six pieces (three line segments and three angles), but some notion of constancy of shape in triangles is important in our geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

**Axiom 5** (SAS Congruence). *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we can take them as axioms and continue.

**Axiom 6** (ASA Congruence). *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

**Axiom 7** (SSS Congruence). *If two triangles have three sides congruent, then the triangles are congruent.*

## 6. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

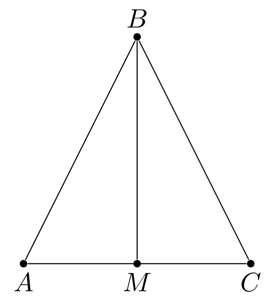
**Theorem 10** (Base angles equal). *If $\triangle ABC$ is isosceles, with base $AC$, then $m\angle A = m\angle C$.*
*Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base $AC$.*

*Proof.* A proof is left as homework. You would want to prove that $\triangle ABC \cong \triangle CBA$. In one case, you would use SSS and in another SAS axioms. □

In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from $A$ is perpendicular to $BC$ (it exists and is unique by Theorem about the existence of the perpendicular); the median from $A$ bisects $BC$ (that is, it crosses $BC$ at a point $D$ which is the midpoint of $BC$); and the angle bisector bisects $\angle A$ (that is, if $E$ is the point where the angle bisector meets $BC$, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

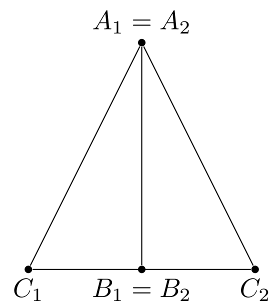
**Theorem 11.** *If $B$ is the apex of the isosceles triangle $ABC$, and $BM$ is the median, then $BM$ is also the altitude, and is also the angle bisector, from $B$.*




*Proof.* Consider triangles $\triangle ABM$ and $\triangle CBM$. Then $AB = CB$ (by definition of isosceles triangle), $AM = CM$ (by definition of midpoint), and side $BM$ is the same in both triangles. Thus, by SAS axiom, $\triangle ABM \cong \triangle CBM$. Therefore, $m\angle ABM = m\angle CBM$, so $BM$ is the angle bisector.
Also, $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = m\angle AMC = 180^\circ$. Thus, $m\angle AMB = m\angle CMB = 180^\circ/2 = 90^\circ$. □

Based on the properties above, we can prove that for right-angle triangle, a hypothenuse and a leg congruence leads to congruence of triangles.

**Theorem 12.** *Let $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ be two right-angle triangles with $m\angle B_1 = m\angle B_2 = 90°$. If $A_1B_1 = A_2B_2$ and $A_1C_1 = A_2C_2$, then $\triangle A_1B_1C_1 \cong \triangle A_2B_2C_2$.*




*Proof.* Take a triangle $\triangle A_2B_2C_2$ and arrange it next to $\triangle A_1B_1C_1$ so that $A_1B_1$ correspond to $A_2B_2$, as on the picture. Note, that since $\angle A_1B_1C_1 = \angle A_2B_2C_2 = 90°$, $C_1C_2$ will be a straight line. Since $A_1C_1 = A_2C_2$, triangle $\triangle A_1C_1C_2$ is isosceles. Therefore, $A_1B_1$ (same as $A_2B_2$) is an altitude of this triangle, and by Theorem 11, it is also a median. Therefore, $C_1B_1 = C_2B_2$. Therefore, triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ are congruent by SSS. □

## 7. Perpendicular Bisector

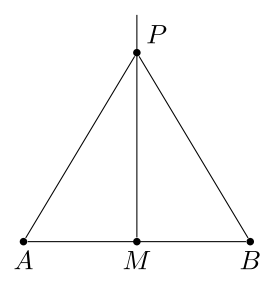
Consider any property of points on the plane — for example, the property that a point $P$ is a distance exactly $r$ from a given point $O$. The set of all points $P$ for which this property holds true is called the locus of points satisfying this property. As we have seen above, the locus of points that are a distance $r$ from a point $O$ is called a circle (specifically, a circle of radius $r$ centered at $O$).
Now consider we are given two points $A$, $B$. If a point $P$ is an equal distance from $A$, $B$ (i.e., if $\overline{PA} \cong \overline{PB}$) then we say $P$ is equidistant from points $A$, $B$.

**Theorem 13.** *The locus of points equidistant from a pair of points $A$, $B$ is a line $l$ which perpendicular to $\overline{AB}$ and goes through the midpoint of $AB$. This line is called the perpendicular bisector of $\overline{AB}$.*

*Proof.* Let $M$ be the midpoint of $\overline{AB}$, and let $l$ be the line through $M$ which is perpendicular to $AB$. We need to prove that for any point $P$,

$$(AP \cong BP) \leftrightarrow P \in l$$

1. Assume that $AP \cong BP$. Then triangle $APB$ is isosceles; by Theorem 11, it implies that $PM \perp AB$. Thus, $PM$ must coincide with $l$, i.e. $P \in l$. Therefore, we have proved implication one way: if $AP \cong BP$, then $P \in l$.
2. Conversly, assume $P \in l$. Then $m\angle AMP = m\angle BMP = 90°$; thus, triangles $\triangle AMP$ and $\triangle BMP$ are congruent by SAS, and therefore $AP \cong BP$.




□

**Theorem 14.** *In a triangle $\triangle ABC$, the perpendicular bisectors of the 3 sides intersect at a single point. This point is the center of a circle circumscribed about the triangle (i.e., such that all three vertices of the triangle are on the circle).*

*Proof.* Consider two perpendicular bisectors to $BC$ and to $AC$. Points on the first one are equidistant from $B$ and $C$; points on the second one and equidistant from $A$ and $C$. Then their intersection point $M$ is thus equidistant from all three $A$, $B$, $C$, and it means that $M$ lies on the perpendicular bisector to $AB$, and it means that all perpendicular bisectors intersect at this point $M$. ☐

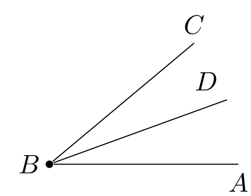
## 8. Median, Altitude, Angle Bisector

Last week we defined three special lines that can be constructed from any vertex in any triangle; each line goes from a vertex of the triangle to the line containing the triangle's opposite side (altitudes may sometimes land on the opposite side outside of the triangle).
Given a triangle $\triangle ABC$,

- The altitude from $A$ is the line through $A$ perpendicular to $\overleftrightarrow{BC}$;
- The median from $A$ is the line from $A$ to the midpoint $D$ of $\overline{BC}$;
- The angle bisector from $A$ is the line $\overleftrightarrow{AE}$ such that $\angle BAE \cong \angle CAE$. Here we let $E$ denote the intersection of the angle bisector with $\overline{BC}$.

The following result is an analog of Theorem 13. For a point $P$ and a line $l$, we define the distance from $P$ to $l$ to be the length of the perpendicular dropped from $P$ to $l$ (see problem 1 in the HW). We say that point $P$ is equidistant from two lines $l$, $m$ if the distance from $P$ to $l$ is equal to the distance from $P$ to $m$.

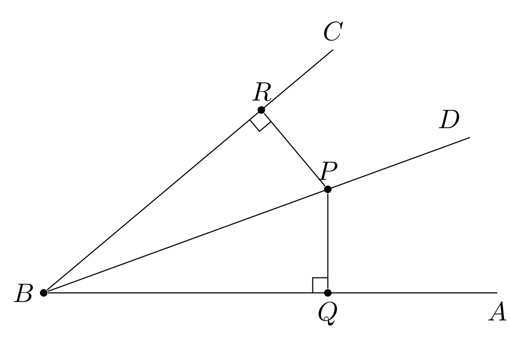
**Theorem 15.** *For an angle $ABC$, the locus of points inside the angle which are equidistant from the two sides $BA$, $BC$ is the ray $\overrightarrow{BD}$ which is the angle bisector of $\angle ABC$.*




*Proof.* We need to prove two things: first, if the point is on a bisector, then it's equidistant from the angle sides; and second, if the point is equidistant from angle sides, it is on the bisector.

**Direction 1:** Assume that the point $P$ is on the bisector $BD$, and $Q$ and $R$ are bases of perpendiculars from $P$ to $AB$ and $CB$ respectively. Then $\triangle PBQ \cong \triangle PBR$ by ASA: $\angle PBR = \angle PBQ$ since $PB$ is a bisector, $PB$ is a common side of these two triangles, and $\angle BQR = \angle BPR$, since sum of angles in a triangle is $90°$. Hence, $PQ = PR$.

**Direction 2:** No assume that $P$ is inside the angle and $PQ = PR$, where $Q$ and $R$ are bases of perpendiculars from $P$ to $AB$ and $CB$ respectively. Then $\triangle PQB$ and $\triangle PRB$ are right-angle triangles, $PQ = PR$, and $PB$ is a common side. By Theorem 12, $\triangle PQB \cong \triangle PRB$, and therefore, $\angle PBQ = \angle PBR$, and point $P$ is on a bisector $PB$. ☐




## 9. Constructions with ruler and compass

Large part of classical geometry are geometric constructions: can we construct a figure with given properties? Traditionally, such constructions are done using straight-edge and compass: the straight-edge tool constructs lines and the compass tool constructs circles. More precisely, it means that we allow the following basic operations:

- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).

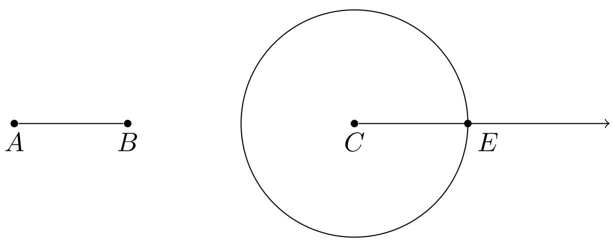
- Draw (construct) a circle with center at previously constructed point $O$ and with radius equal to distance between two previously constructed points $B, C$
- Construct the intersections point(s) of two previously constructed lines, circles, or a circle and a line

All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!

Constructions of this form have been famous since mathematics in ancient Greece. Here are some examples of constructions:

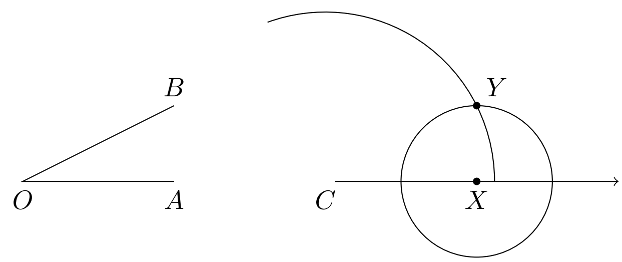
**Example 1.** Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point $E$ on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

*Construction.* Construct a circle centered at $C$ with radius $AB$. Then this circle will intersect $\overrightarrow{CD}$ at the desired point $E$. □

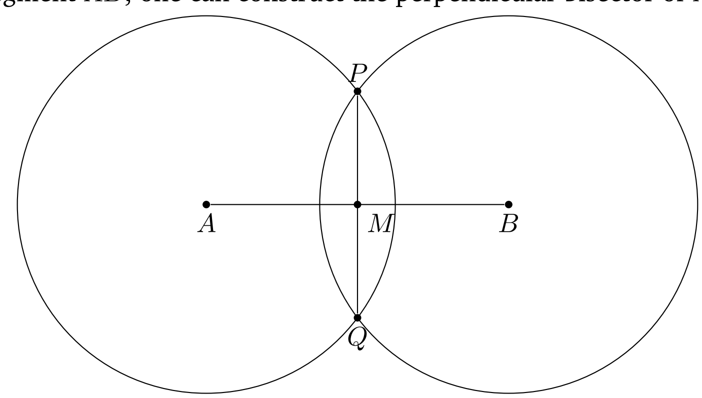



**Example 2.** Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

*Construction.* First construct point $X$ on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius $OB$ centered at $C$ and a circle of radius $AB$ centered at $X$. Let $Y$ be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □




**Example 3.** Given a segment $AB$, one can construct the perpendicular bisector of $AB$. Here is the picture:




A great tool to learn these constructions is an app called *Euclidea*. You can use it in a web browser at http://euclidea.xyz, or install it on your phone or tablet (it is available both for iOS and Android).

Note: Euclidea starts with a slightly more restrictive set of tools. Namely, it only allows drawing circles with a given center and passing through a given point; thus, you can not use another segment as radius.

## 10. Triangle inequalities

In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal. The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must be longer than the one opposite the smaller angle.

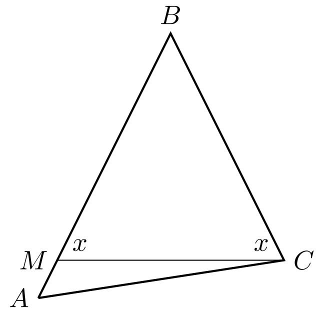
**Theorem 16.** *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

*Proof.* Assume by contradiction that it is not the case. Then either $BC = AB$ or $BC < AB$.

But if $BC = AB$, then $\triangle ABC$ is isosceles, so by Theorem 10, $m\angle A = m\angle C$ as base angles, which gives a contradiction.

Now assume $BC < AB$, find the point $M$ on $AB$ so that $BM = BC$, and draw the line $MC$. Then $\triangle MBC$ is isosceles, with apex at $B$. Hence $m\angle BMC = m\angle MCB$ (these two angles are denoted by $x$ in the figure.) On one hand, $m\angle C > x$ (this easily follows from Axiom 3). On the other hand, since $x$ is an external angle of $\triangle AMC$, we have $x > m\angle A$ (by External Angle Theorem 9). These two inequalities imply $m\angle C > m\angle A$, which contradicts what we started with.

Thus, assumptions $BC = AB$ or $BC < AB$ both lead to a contradiction.




□

The converse of the previous theorem is also true: opposite a longer side, there must be a larger angle.The proof is left as an exercise.

**Theorem 17** (Slant lines and perpendiculars)**.** *Let $P$ be a point not on line $l$, and let $Q \in l$ be such that $PQ \perp l$. Then for any other point $R$ on line $l$, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.*

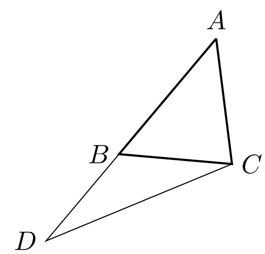
*Proof.* The proof easily follows from Theorem 16 since in the triangle $\triangle PQR$ the angle opposite of a slant line is $90°$ and is larger than the angle opposite the perpendicular. □

**Theorem 18.** *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

The following theorem doesn't quite say that a straight line is the shortest distance between two points, but it says something along these lines. This result is used throughout much of mathematics, and is referred to as "the triangle inequality".

**Theorem 19** (The triangle inequality)**.** *In $\triangle ABC$, we have $AB + BC > AC$.*

*Proof.* Extend the line $AB$ past $B$ to the point $D$ so that $BD = BC$, and join the points $C$ and $D$ with a line so as to form the triangle $ADC$. Observe that $\triangle BCD$ is isosceles, with apex at $B$; hence $m\angle BDC = m\angle BCD$. It is immediate that $m\angle DCB < m\angle DCA$. Looking at $\triangle ADC$, it follows that $m\angle D < m\angle C$; by Theorem 16, this implies $AD > AC$. Our result now follows from $AD = AB + BD$ (Axiom 2) □




## 11. Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex $A$ is opposite vertex $C$). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex $A$ is opposite vertex $C$, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

A quadrilateral is called

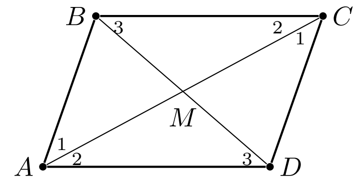
- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

**Theorem 20.** *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$

- *The intersection point $M$ of diagonals $AC$ and $BD$ bisects each of them.*




*Proof.* Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 about alternate interior angles, angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

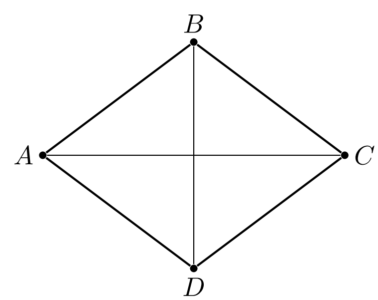
Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. □

**Theorem 21.** *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*

Proof is left to you as a homework exercise. In fact, one can prove that all of the properties of parallelogram are equivalent, and any of them can potentially be taken as a definition of a parallelogram:

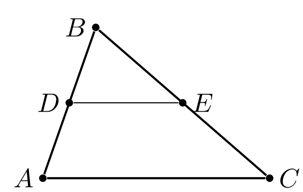
1. Opposite sides are parallel;
2. Opposite sides are equal;
3. Opposite angles are equal;
4. One pair of opposite sides is parallel and equal;
5. Diagonals bisect each other.

**Theorem 22.** *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*




*Proof.* Since the opposite sides of a rhombus are equal, it follows from Theorem 21 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let $M$ be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and $BM$ is a median, by Theorem 11, it is also the altitude. □

## 12. Midline of a triangle and a trapezoid

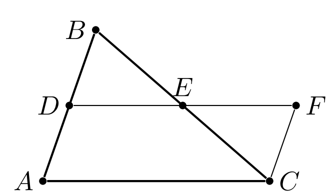



A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.

**Theorem 23.** *If $DE$ is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

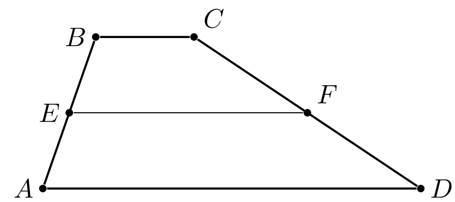
*Proof.* Continue line $DE$ and mark on it point $F$ such that $DE = EF$.

1. $\triangle DEB \cong \triangle FEC$ by SAS: $DE = EF$, $BE = EC$, $\angle BED \cong \angle CEF$.
2. $ADFC$ is a parallelogram: First, we can see that since $\triangle DEB \cong \triangle FEC$, then $\angle BDE \cong \angle CFE$, and since they are alternate interior angles, $AD \parallel FC$. Also, from the same congruency, $FC = BD$, but $BD = AD$ since $D$ is a midpoint. Then, $FC = DA$. So we have $FC = DA$ and $FC \parallel DA$, and therefore $ADFC$ is a parallelogram.
3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$.

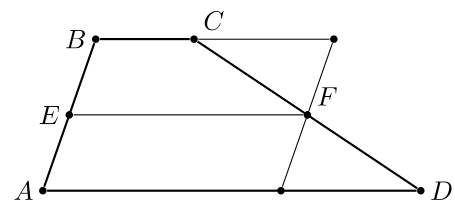



□

**Theorem 24** (Trapezoid midline)**.** *Let $ABCD$ be a trapezoid, with bases $AD$ and $BC$, and let $E$, $F$ be midpoints of sides $AB$, $CD$ respectively. Then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.*




**Idea of the proof:** draw through point $F$ a line parallel to $AB$, as shown in the figure. Prove that this gives a parallelogram, in which points $E$, $F$ are midpoints of opposite sides.




## 13. Circles

A circle with center $O$ and radius $r > 0$ is the set of all points $P$ in the plane such that $OP = r$. Traditionally, one denotes circles by Greek letters: $\lambda, \omega \dots$.

Given a circle $\lambda$ with center $O$,

- A radius is any line segment from $O$ to a point $A$ on $\lambda$,
- A chord is any line segment between distinct points $A$, $B$ on $\lambda$,
- A diameter is a chord that passes through $O$,

Recall that by Theorem 13, if $O$ is equidistant from points $A, B$, then $O$ must lie on the perpendicular bisector of $AB$. We can restate this result as follows.

**Theorem 25.** *If $AB$ is a chord of circle $\lambda$, then the center $O$ of this circle lies on the perpendicular bisector of $AB$.*

## 14. Relative positions of lines and circles

**Theorem 26.** *Let $\lambda$ be a circle of radius $r$ with center at $O$ and let $l$ be a line. Let $d$ be the distance from $O$ to $l$, i.e. the length of the perpendicular $OP$ from $O$ to $l$. Then:*

- *If $d > r$, then $\lambda$ and $l$ do not intersect.*
- *If $d = r$, then $\lambda$ intersects $l$ at exactly one point $P$, the base of the perpendicular from $O$ to $l$. In this case, we say that $l$ is tangent to $\lambda$ at $P$.*
- *If $d < r$, then $\lambda$ intersects $l$ at two distinct points.*

*Proof.* First two parts easily follow from Theorem 17: slant line is longer than the perpendicular.

For the last part, it is easy to show that $\lambda$ can not intersect $l$ at more than 2 points. Proving that it does intersect $l$ at two points is very hard and requires deep results about real numbers. This proof will not be given here. $\square$

Note that it follows from the definition that a tangent line is perpendicular to the radius $OP$ at point of tangency. Converse is also true.

**Theorem 27.** *Let $\lambda$ be a circle with center $O$, and let $l$ be a line through a point $A$ on $\lambda$. Then $l$ is tangent to $\lambda$ if and only if $l \perp \overleftrightarrow{OA}$*

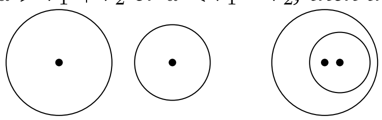
*Proof.* By definition, if $l$ is the tangent line to $\lambda$, then it has only one common point with $\lambda$, and this point is the base of the perpendicular from $O$ to $l$; thus, $OA$ is the perpendicular to $l$.

Conversely, if $OA \perp l$, it means that the distance from $l$ to $O$ is equal to the radius (both are given by $OA$), so $l$ is tangent to $\lambda$. $\square$

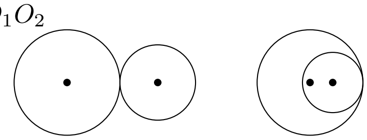
Similar results hold for relative position of a pair of circles. We will only give part of the statement.

**Theorem 28.** *Let $\lambda_1, \lambda_2$ be two circles, with centers $O_1, O_2$ and radiuses $r_1, r_2$ respectively; assume that $r_1 \geq r_2$. Let $d = O_1O_2$ be the distance between the centers of the two circles.*

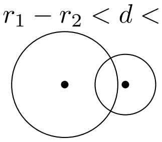
- *If $d > r_1 + r_2$ or $d < r_1 - r_2$, then these two circles do not intersect.*



- *If $d = r_1 + r_2$ or $d = r_1 - r_2$ then these two circles have a unique common point, which lies on the line $O_1O_2$*



- *If $r_1 - r_2 < d < r_1 + r_2$, then the two circles intersect at exactly two points.*



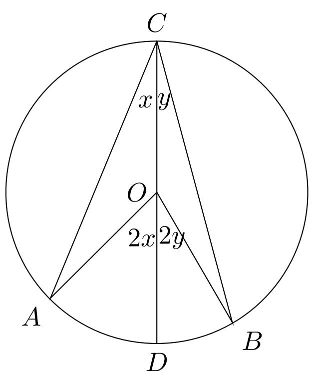
We skip the proof.

**Definition.** Two circles are called tangent if they intersect at exactly one point.

## 15. Arcs and Angles

Consider a circle $\lambda$ with center $O$, and an angle formed by two rays from $O$. Then these two rays intersect the circle at points $A$, $B$, and the portion of the circle contained inside this angle is called the arc subtended by $\angle AOB$. We will sometimes use the notation $\overset{\frown}{AB}$. We define the measure of the arc as the measure of the corresponding central angle: $\overset{\frown}{AB} = m\angle AOB$.

**Theorem 29.** *Let $A$, $B$, $C$ be on circle $\lambda$ with center $O$. Then $m\angle ACB = \frac{1}{2}\overset{\frown}{AB}$. The angle $\angle ACB$ is said to be inscribed in $\lambda$.*




*Proof.* There are actually a few cases to consider here, since $C$ may be positioned such that $O$ is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.
*Case 1.* Draw diameter $\overline{CD}$. Let $x = m\angle ACD$, $y = m\angle BCD$, so that $m\angle ACB = x + y$.
Since $\overline{OC}$ is a radius of $\lambda$, we have that $\triangle AOC$ is isosceles triangle, thus $m\angle A = x$. Therefore, $m\angle AOD = 2x$, as it is the external angle of $\triangle AOC$. Similarly, $m\angle BOD = 2y$. Thus, $\overset{\frown}{AB} = \overset{\frown}{AD} + \overset{\frown}{DB} = 2x + 2y$. □

This theorem has a converse, which essentially says that **all** points $C$ forming a given angle $\angle ACB$ with given points $A, B$ must lie on a circle containing points $A, B$. Exact statement is given in the homework (see problem 38).

As an immediate corollary, we get the following result:

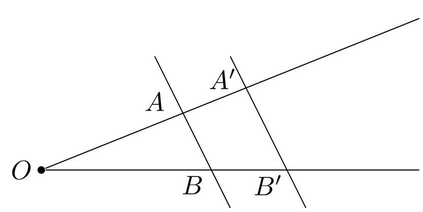
**Theorem 30.** *Let $\lambda$ be a circle with diameter $AB$. Then for any point $C$ on this circle other than $A, B$, the angle $\angle ACB$ is the right angle. Conversely, if a point $C$ is such that $\angle ACB$ is the right angle, then $C$ must lie on the circle $\lambda$.*

## 16. Thales Theorem

**Theorem 31** (Thales Theorem)**.** *Let points $A'$, $B'$ be on the sides of angle $\angle AOB$ as shown in the picture. Then lines $AB$ and $A'B'$ are parallel if and only if*

$$\frac{OA}{OB} = \frac{OA'}{OB'}$$

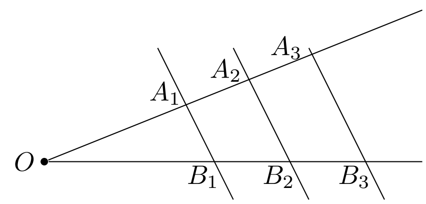
*In this case, we also have $\frac{OA}{OB} = \frac{AA'}{BB'}$*



We have already seen and proved a special case of this theorem when discussing the midline of a triangle.

The proof of this theorem is unexpectedly hard. In the case when $\frac{OA}{OA'}$ is a rational number, one can use arguments similar to those we did when talking about midline. The case of irrational numbers is harder yet. We skip the proof for now; it will be discussed in Math 9.

As an immediate corollary of this theorem, we get the following result.




**Theorem 32.** *Let points $A_1, \dots, A_n$ and $B_1, \dots B_n$ on the sides of an angle be chosen so that $A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n$, and lines $A_1B_1$, $A_2B_2$, ... are parallel. Then $B_1B_2 = B_2B_3 = \dots = B_{n-1}B_n$.*

Proof of this theorem is left to you as exercise.

## 17. Similar triangles

**Definition.** Two triangles $\triangle ABC$, $\triangle A'B'C'$ are called *similar* if

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

and the corresponding sides are proportional, i.e.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

The common ratio $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ is sometimes called the similarity coefficient.

There are some similarity tests:

**Theorem 33** (AAA similarity test)**.** *If the corresponding angles of triangles $\triangle ABC$, $\triangle A'B'C'$ are equal:*

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

*then the triangles are similar.*

**Theorem 34** (SSS similarity test)**.** *If the corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

*then the triangles are similar.*

**Theorem 35** (SAS similarity test)**.** *If two pairs of corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

*and $\angle A \cong \angle A'$ then the triangles are similar.*

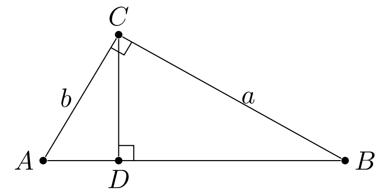
Proofs of all of these tests can be obtained from Thales theorem.

## 18. Pythagorean Theorem

Pythagorean theorem has dozens of proofs, but most of them are based on the notion of area and thus can't be considered completely rigorous until we give a definition of area and prove properties we normally take for granted, such as that when we cut a figure in two, the area of the original figure is equal to sum of areas of the pieces. These are actually rather hard to prove. The proof below, based on similar triangles, is one of the shortest *rigorous* proofs.

**Theorem 36** (Pythagorean theorem)**.** *Let $ABC$ be a right triangle, $\angle C = 90°$, and let $CD$ be the altitude. Denote by $a, b, c$ the lengths of the sides of this triangle: $a = BC$, $b = AC$, $c = AB$. Then*

$$c^2 = a^2 + b^2$$




*Proof.* You can complete the proof yourself by following the following logic

**1.** Show that triangles $\triangle ACD$, $\triangle ABC$ are similar and deduce from this that $AD = b^2/c$.
**2.** Similarly, prove that $BD = a^2/c$
**3.** Now, since $AB = c = AD + DB$, prove $c^2 = a^2 + b^2$.

□

HOMEWORK

**1.** It is important that you know some geometry notation.
   (a) What does the symbol $\parallel$ mean? How do you pronounce it? How would you read "$a \parallel b$"?
   (b) What does the symbol $\perp$ mean? How would you say "$a \perp b$"?
   (c) Suppose you have two points $X$ and $Y$. What is the difference between $\overline{XY}$, $\overleftrightarrow{XY}$, $\overrightarrow{XY}$? What are each of these things called?
   (d) Given three points $E$, $F$, $G$, what does $EF + FG$ mean?
   (e) Given four points $A$, $B$, $C$, $D$, what does $m\angle ADC + m\angle BDC$ mean? If I tell you $m\angle ADC + m\angle BDC = 180°$, does that tell you any information about $m\angle ADC$ or $m\angle BDC$?
   (f) What does the symbol $\triangle$ mean? For example, if $A$ and $B$ and $C$ are points, what is $\triangle ABC$?

**2.** (a) What is a proof? Give an example. Can you come up with an example that is not about geometry?
   (b) What is an axiom? Give an example. Can you come up with an example that is not about geometry?

**3.** In this problem, you will make diagrams. Part of the purpose of this exercise is so that, when you think about geometry, the pictures in your notes or in your mind aren't all just the diagrams I draw out for you in class or on classwork sheets. You have to be able to draw or visualize configurations of lines other than the way they're set up in axiom 4, for example.
   (a) Given lines $a, b, c$, is it possible that $a \parallel b$ and $\neg(b \parallel c)$ but $a \parallel c$? Draw a diagram and then explain your reasoning on how to answer this question. ("explain" means, of course, in writing.)
   (b) Suppose we have parallel lines $l, m$. Let $A, B, C$ be points on $l$, with $B$ between $A, C$. Let $X, Y, Z$ be points on $m$, with $Y$ between $X, Z$. Is it possible for lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, $\overleftrightarrow{CZ}$ to all intersect at one point? Draw a diagram of what this might look like.
   (c) Consider the diagram you drew in the previous part, with the lines $l, m$ and the six points, and the three cross-lines that intersect at a point. Now consider the lines $\overleftrightarrow{AZ}$, $\overleftrightarrow{CX}$. Do these two lines intersect at a point on $\overleftrightarrow{BY}$? Draw a diagram where this *is* the case, and then draw a second diagram where this *is not* the case.
   (d) Draw a rectangle that's not a square, and draw it so that one of the bases is horizontal. Then draw one of the rectangle's diagonals. Notice that, of the two right angles formed at the rectangle's base, the rectangle's diagonal splits one of those angles into two smaller angles. Which of the two angles is bigger - the one below the diagonal, or the one above the diagonal? Draw a second rectangle where the opposite relation holds true (for example, if the lower angle was bigger in your first rectangle, draw a second rectangle where the lower angle split by the diagonal is smaller).

**4.** Can you formulate Axiom 4 without referring to the picture (i.e. without using any statement such as "angles $\angle 1$, $\angle 2$ are as shown in the figure below"? You will have to introduce a number of points and have very clear notations.

**5.** The following logic and geometric statements come in equivalent pairs. Each logic statement has exactly one geometric statement that is equivalent to it. Match these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent. [**Note**: the quantifier $\exists!$ stands for "there exists a unique…", and $\emptyset$ is an empty set.]

**Geometric statements:**

(a) For any two distinct points there is a unique line containing these points.
(b) Given a line and a point not on the line there exists a unique line though the given point that is parallel to the given line.
(c) If two lines are parallel and another line intersects one of them, then it intersects the other one as well.
(d) If two lines are parallel to the same line, then they are parallel to each other

**Logic statements:**

(a) $\forall l\ \forall m$ such that $l \parallel m\ [\forall n\ (n \cap l \neq \emptyset \to n \cap m \neq \emptyset)]$
(b) $\forall A\ \forall B$ such that $A \neq B \quad [\exists!\ l\ (A \in l \wedge B \in l)]$
(c) $\forall l\ \forall m\ [(\exists n$ such that $n \parallel l \wedge n \parallel m) \to (l \parallel m)]$
(d) $\forall l\ \forall A$ such that $A \notin l \quad [\exists!\ m\ (A \in m \wedge m \parallel l)]$

**6.** (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 4 will be of key importance.
   (a) Study the proof of Theorem 2 and draw a diagram that illustrates it.
   (b) Study the proof of Theorem 3.
   (c) Prove Theorem 5.
   (d) Prove Theorem 6.
   (e) Prove Theorem 7.

**7.** Complete the proof of Theorem 8, about sum of angles of a triangle.

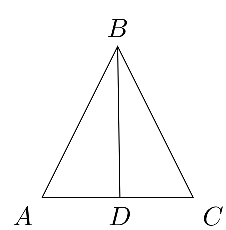
**8.** What is the sum of angles of a quadrilateral? of a pentagon?

**9.** Notice that SSA and AAA are not listed as congruence rules.
   (a) Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.
   (b) Describe a pair of triangles that have three congruent angles but are not congruent triangles.

**10.** Prove that the following two properties of a triangle are equivalent:
   (a) All sides have the same length.
   (b) All angles are $60°$.

   A triangle satisfying these properties is called *equilateral*.

**11.** A triangle in which two sides are congruent is called *isosceles*. Such triangles have many special properties.
   (a) Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose $D$ is a point on $\overline{AC}$ such that $\overline{AD} \cong \overline{DC}$ (such point is called *midpoint* of the segment). Prove that then, $\triangle BD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?
   (b) Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.




**12.** (Slant lines and perpendiculars) Let $P$ be a point not on line $l$, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point $R$ on line $l$, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line.

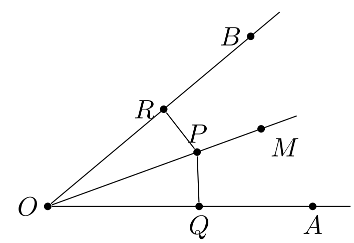
   **Note**: you can not use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 16.

**13.** (Angle bisector). Define a distance from a point $P$ to line $l$ as the length of the perpendicular from $P$ to $l$ (compare with the previous problem).

Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i.e. $\angle AOM \cong \angle MOB$.

(a) Let $P$ be any point on $\overrightarrow{OM}$, and $PQ$, $PR$ – perpendiculars from $P$ to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$, $\triangle OPQ$ are congruent, and deduce from this that distances from $P$ to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.

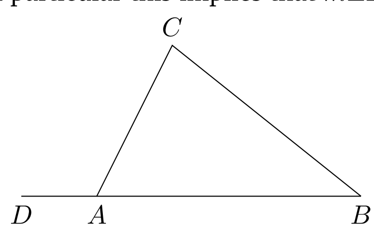
(b) Prove that conversely, if $P$ is a point inside angle $\angle AOB$, and distances from $P$ to the two sides of the angle are equal, then $P$ must lie on the angle bisector $\overrightarrow{OM}$




These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector

**14.** Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors)

**15.** Given a triangle $\triangle ABC$, let $D$ be a point on the line $AB$, so that $A$ is between $D$ and $B$. In this situation, angle $\angle DAC$ is called an *external angle* of $\triangle ABC$. Prove that $m\angle DAC = m\angle B + m\angle C$ (in particular this implies that $m\angle DAC > m\angle B$, and similarly for $\angle C$).




**16.** (Perpendicular bisector) Let $\overline{AB}$ be a line segment. The perpendicular bisector $L$ of $\overline{AB}$ is the line that passes through the midpoint $M$ of $\overline{AB}$ and is perpendicular to $\overline{AB}$.

(a) Prove that for any point $P$ on $L$, triangles $\triangle APM$ and $\triangle BPM$ are congruent. Deduce from this that $AP = BP$.

(b) Conversely, let $P$ be a point on the plane such that $AP = BP$. Prove that then $P$ must be on $L$.

Taken together, these two statements say that a point $P$ is *equidistant* from $A, B$ if and only if it lies on the perpendicular bisector $L$ of segment $\overline{AB}$. Another way to say it is that the *locus* of all the points equidistant from $A, B$ is the perpendicular bisector of $\overline{AB}$.

**17.** Show that for any triangle $\triangle ABC$, the perpendicular bisectors of the three sides intersect at a single point, and this point is equidistant from all three vertices of the triangle. [**Hint**: consider the point where two of the bisectors intersect. Prove that this point is equidistant from all three vertices.]

**Note**: the intersection point can be outside the triangle.

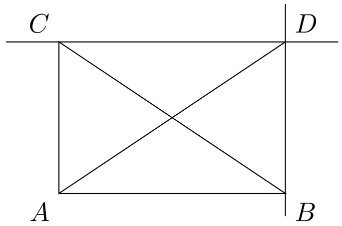
**18.** Let $P$ be a point not on line $l$, and $A \in l$ be the base of perpendicular from $P$ to $l$: $AP \perp l$. Prove that for any other point $B$ on $l$, $PB > PA$ ("perpendicular is the shortest distance"). Note: you can not use Pythagorean theorem as we have not proved it yet; instead, try using Theorem 11 (opposite the larger angle there is a longer side).

**19.** Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let $D$ be the intersection of the line parallel to $\overline{AB}$ through C with the line parallel to $\overline{AC}$ through B.

(a) Prove $\triangle ABC \cong \triangle DCB$

(b) Prove $\triangle ABC \cong \triangle BDA$

(c) Prove that $\overline{AD}$ is a median of $\triangle ABC$.



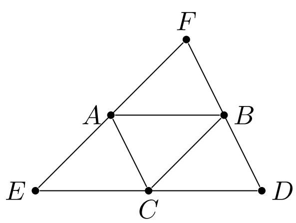
**20.** Let $\triangle ABC$ be a right triangle with right angle $\angle A$, and let $D$ be the midpoint of $\overline{BC}$. Prove that $AD = \frac{1}{2}BC$.

**21.** Let $l_1$, $l_2$ be the perpendicular bisectors of side $AB$ and $BC$ respectively of $\triangle ABC$, and let $F$ be the intersection point of $l_1$ and $l_2$. Prove that then $F$ also lies on the perpendicular bisector of the side $BC$. [Hint: use Theorem 14.]

**22.** Prove Theorem 15.

**23.** Let the angle bisectors from $B$ and $C$ in the triangle $\triangle ABC$ intersect each other at point $F$. Prove that $\overleftrightarrow{AF}$ is the third angle bisector of $\triangle ABC$. [Hint: use Theorem 16]

**24.** Given triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the vertices of the resulting triangle by $D$, $E$, $F$, as shown in the figure below.



(a) Prove that $\triangle ABC \cong \triangle BAF$ (pay attention to the order of vertices). Similarly one proves that all four small triangles in the picture are congruent.
(b) Prove that $\overline{AB} \parallel \overline{ED}$ and $AB = \frac{1}{2}ED$.
(c) Prove that perpendicular bisectors of sides of $\triangle DEF$ are altitudes of $\triangle ABC$.
(d) Show that in any triangle, the three altitudes meet at a single point.

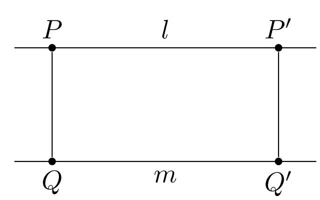
**25.** (Parallelogram) Who doesn't love parallelograms?
(a) Prove Theorem 21.
(b) Prove that if in a quadrilateral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

**26.** Prove that in a parallelogram, sum of two adjacent angles is equal to $180°$:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°$$

**27.** (Rectangle) A quadrilateral is called rectangle if all angles have measure $90°$.
(a) Show that each rectangle is a parallelogram.
(b) Show that opposite sides of a rectangle are congruent.
(c) Prove that the diagonals of a rectangle are congruent.
(d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.

**28.** (Distance between parallel lines)
Let $l, m$ be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $\overleftrightarrow{PQ} \perp l$ (by Theorem 6, this implies that $\overleftrightarrow{PQ} \perp m$).
Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $\overleftrightarrow{P'Q'} \perp l$, we have $PQ = P'Q'$. (This common distance is called the distance between $l$, $m$.)



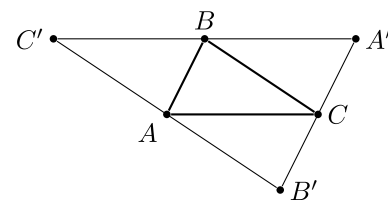
**29.** The following statements about a parallelogram can be used as its definition, i.e. you can prove any of them from any other. Can you show how?

We have done some of the proofs already. Establish which other statements need to be proven to show the equivalence of all of these statements, and try to prove them. For example, Theorem 21 proves (b) $\Rightarrow$ (a), and Theorem 20 proves (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), and (a) $\Rightarrow$ (d); (e) $\Rightarrow$ (a) is proven in Problem 4b.

(a) Opposite sides are parallel.
(b) Opposite sides are congruent.
(c) Opposite angles are congruent.
(d) Diagonals bisect each other.
(e) One pair of opposite sides is parallel and congruent.

**30.** Show that if we mark midpoints of each of the three sides of a triangle, and connect these points, the resulting segments will divide the original triangle into four triangles, all congruent to each other.

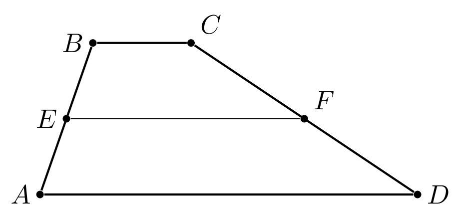
**31.** (Altitudes intersect at single point)
The goal of this problem is to prove that three altitudes of a triangle intersect at a single point. Given a triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the intersection points of these lines by $A', B', C'$ as shown in the figure.



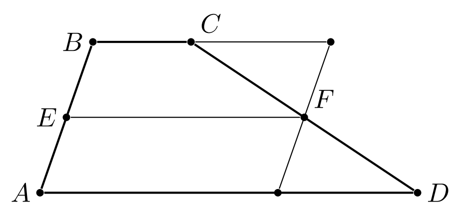

(a) Prove that $A'B = AC$ (hint: use parallelograms!)
(b) Show that $B$ is the midpoint of $A'C'$, and similarly for other two vertices.
(c) Show that altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of sides of $\triangle A'B'c'$.
(d) Prove that the three altitudes of $\triangle ABC$ intersect at a single point.

**32.** (Trapezoid Midline)

Let $ABCD$ be a trapezoid, with bases $AD$ and $BC$, and let $E$, $F$ be midpoints of sides $AB$, $CD$ respectively. Prove that then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.




[Hint: draw through point $F$ a line parallel to $AB$, as shown in the figure. Prove that this gives a parallelogram, in which points $E$, $F$ are midpoints of opposite sides. ]




**33.** Without using Theorem 26, prove that a circle can not have more than two intersections with a line. [Hint: assume it has three intersection points, and use Theorem 25 to get a contradiction.]

**34.** Prove that given three points $A, B, C$ not on the same line, there is a unique circle passing through these points. This circle is called the circumscribed circle of $\triangle ABC$. Explain how to construct this circle using ruler and compass.

**35.** Show that if a circle $\omega$ is tangent to both sides of the angle $\angle ABC$, then the center of that circle must lie on the angle bisector. [Hint: this center is equidistant from the two sides of the circle.] Show that conversely, given a point $O$ on the angle bisector, there exists a circle with center at this point which is tangent to both sides fo the angle.

**36.** Use the previous problem to show that for any triangle, there is a unique circle that is tangent to all three sides (inscribed circle).

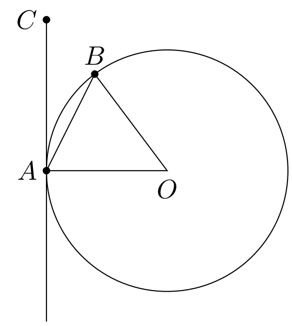
**37.** Given a circle $\lambda$ with center $A$ and a point $B$ outside this circle, construct the tangent line $l$ from $B$ to $\lambda$ using straightedge and compass. How many solutions does this problem have?

[Hint: let $P$ be the tangency point (which we haven't contructed yet). Then by Theorem 27, $\angle APB$ is a right angle. Thus, by Theorem 30, it must lie on a circle with diameter $OP$]

**38.** (Angle Theorems) Let's study Theorem 29 in a bit more detail!

(a) Prove the converse of Theorem 29: namely, if $\lambda$ is a circle centered at $O$ and $A$, $B$, are on $\lambda$, and there is a point $C$ such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then $C$ lies on $\lambda$. [Hint: let $C'$ be the point where line $AC$ intersects $\lambda$. Show that then, $m\angle ACB = m\angle AC'B$, and show that this implies $C = C'$.]

(b) Let $A$, $B$ be on circle $\lambda$ centered at $O$ and $m$ the tangent to $\lambda$ at $A$, as shown on the right. Let $C$ be on $m$ such that $C$ is on the same side of $\overleftrightarrow{OA}$ as $B$. Prove that $m\angle BAC = \frac{1}{2}m\angle BOA$. [Hint: extend $\overline{OA}$ to intersect $\lambda$ at point $D$ so that $\overline{AD}$ is a diameter of $\lambda$. What arc does $\angle DAB$ subtend?]

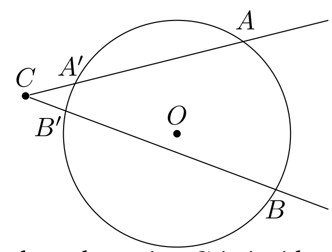



**39.** Here is a modification of Theorem 29.

Consider a circle $\lambda$ and an angle whose vertex $C$ is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$.

Prove that in this case, $m\angle C = \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{A'B'})$.

[Hint: draw line $AB'$ and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]




**40.** Can you suggest and prove an analog of the previous problem, but when the point $C$ is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vertical angles)?

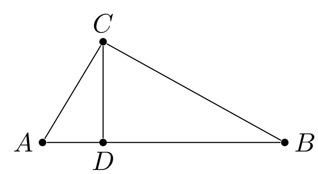
**41.** Complete levels $\alpha$, $\beta$ in Euclidea.

**42.** Prove Theorem 32 (using Thales Theorem). Hint: let $k = \frac{OB_1}{OA_1}$; show that then $B_iB_{i+1} = kA_iA_{i+1}$.

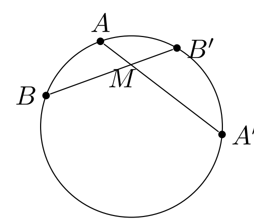
**43.** Using Theorem 32, describe how one can divide a given segment into 5 equal parts using ruler and compass.

**44.** Given segments of length $a$, $b$, $c$, construct a segment of length $\frac{ab}{c}$ using ruler and compass.

**45.** Let $ABC$ be a right triangle, $\angle C = 90°$, and let $CD$ be the altitude. Prove that triangles $\triangle ACD$, $\triangle CBD$ are similar. Deduce from this that $CD^2 = AD \cdot DB$.




**46.** Let $M$ be a point inside a circle and let $AA'$, $BB'$ be two chords through $M$. Show that then $AM \cdot MA' = BM \cdot MB'$. [Hint: use inscribed angle theorem to show that triangles $\triangle AMB$, $\triangle B'MA'$ are similar. ]

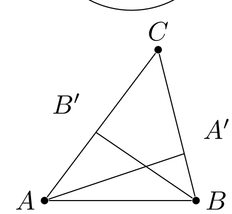



**47.** Let $AA'$, $BB'$ be altitudes in the acute triangle $\triangle ABC$.

(a) Show that points $A'$, $B'$ are on a circle with diameter $AB$.

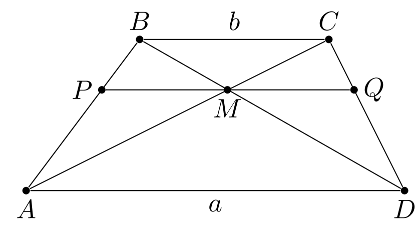
(b) Show that $\angle AA'B' = \angle ABB'$, $\angle A'B'B = \angle A'AB$

(c) Show that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C$.




**48.** Complete the proof of Pythagorean Theorem 36.

**49.** Consider the trapezoid with bases $AD = a$, $BC = b$. Let $M$ be the intersection point of diagonals, and let $PQ$ be the segment parallel to the bases through $M$.

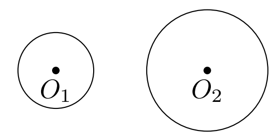



(a) Show that point $M$ divides each of diagonals in proportion $a : b$, e.g. $AM : MC = a : b$.

(b) Show that points $P$, $Q$ divide sides of the trapezoid in proportion $a : b$.

(c) Show that $PQ = \frac{2ab}{a+b}$. [Hint: compute $PM$, $MQ$ separately and add.]

**50.** Given two circles with centers $O_1, O_2$ and radiuses $r_1, r_2$ respectively, construct (using straightedge and compass) a common tangent line to these circles. You can assume that circles do not intersect: $O_1O_2 > r_1 + r_2$ and that $r_2 > r_1$.




[Hint: assume that we have such a tangent line, call it $l$. Then distance from that line to $O_1$ is $r_1$, and distance to $O_2$ is $r_2$. Thus, if we draw a line $l'$ parallel to $l$ but going through $O_1$, the distance from $l'$ to $O_2$ is ... and thus $l'$ is tangent to ...]