

Math 7: Handout 10 [2022/12/04] Binomial Probabilities

CLASSWORK

Binomial coefficients are useful for calculating probabilities. Let us first introduce the terminology:

Trial: one instance of an experiment. For example, if we are doing a sequence of coin tosses, each coin toss is a *trial*. If we are shooting ducks, each shot is a *trial*.

n : Number of trials is denoted by n .

Success: a trial that ends up in a desired outcome. If we are looking for Heads, *success* is an outcome of getting a Head. If we are looking at duck shooting, *success* is a hit.

k : Number of successes we need.

p : Probability of success in one trial ($1/2$ for heads/tails)

Failure: a trial that does not end up in a success (missing a duck, getting a Tail while looking for Heads)

q : Probability of failure, $q = 1 - p$.

Imagine that we want to calculate the probability of getting k successes in n trials, and we know p . Then:

$$P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k q^{n-k}, \text{ where}$$

- p — probability of success in one try;
- $q = 1 - p$ — probability of failure in one try;
- n — number of trials;
- k — number of successes;
- $n - k$ — number of failures.

Example: You roll a 6-sided die 6 times. What is the probability of getting a 6 exactly once? Twice? Three times?

Solution: For a dice, probability $p = 1/6$ (getting six), $q = 5/6$ (getting anything else). Out of $n = 6$ rolls, you will get six with probability

$$k = 1 \text{ times:} \quad P_1 = \binom{6}{1} \cdot \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^5 = 6 \cdot \frac{5^5}{6^6} = \frac{5^5}{6^5} \approx 0.402;$$

$$k = 2 \text{ times:} \quad P_2 = \binom{6}{2} \cdot \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4 = \frac{6 \cdot 5}{1 \cdot 2} \cdot \frac{5^4}{6^6} = \frac{5^5}{2 \cdot 6^5} = 0.201;$$

$$k = 3 \text{ times:} \quad P_3 = \binom{6}{3} \cdot \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3 = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{5^3}{6^6} = \frac{4 \cdot 5^4}{6^6} = 0.0535;$$

Example: A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$. What is the probability that out of 7 shots, she will hit exactly once? twice? three times? not hitting at all?

Solution: Now the probabilities are $p = 1/3$, $q = 2/3$. Out of $n = 7$ shots, probability of hitting is

$$k = 1 \text{ times:} \quad P_1 = \binom{7}{1} \cdot \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{7-1} = 7 \cdot \frac{2^6}{3^7} = 0.2048;$$

$$k = 2 \text{ times:} \quad P_2 = \binom{7}{2} \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{7-2} = \frac{7 \cdot 6}{1 \cdot 2} \cdot \frac{2^5}{3^7} = 0.3073;$$

$$k = 3 \text{ times:} \quad P_3 = \binom{7}{3} \cdot \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{7-3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{2^4}{3^7} = 0.2561;$$

$$\text{not hitting at all} \quad P_0 = \binom{7}{0} \cdot \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{7-0} = 1 \cdot \frac{1}{3^7} = 0.0585.$$

HOMEWORK

In the problems below, you can give your answer as a binomial coefficient without calculating it. If you want to calculate it, use Pascal triangle: $\binom{n}{k}$ is the k -th element in the n -th row of the Pascal triangle, counting from 0.

1. A (blindfolded) marksman finds that on the average he hits the target 4 times out of 5. If he fires 4 shots, what is the probability of
 - (a) more than 2 hits?
 - (b) at least 3 misses?
2. In each of 4 races between Tories and Whigs, the Tories have a 60% chance of winning. Assuming that the races are independent of each other, what is the probability that:
 - (a) The Tories will win 0 races, 1 race, 2 races, 3 races, or all 4 races?
 - (b) The Tories will win at least 1 race.
 - (c) The Tories will win a majority of the races.
3. The ratio of boys to girls at birth in Singapore is quite high at 1.09 : 1. What proportion of Singapore families with exactly 6 children will have at least 3 boys?
4. A bag has 10 red marbles and 15 black marbles. One ball at a time is pulled from a bag, recorded, and then put back in the bag. What is the probability:
 - (a) To pull 5 red marbles in a row?
 - (b) To pull 7 black marbles in a row?
 - (c) To pull exactly 2 red marbles and 4 black marbles in any order?
 - (d) To pull exactly 3 red marbles and 3 black marbles in any order?
5. In a class of 11 students, what is the probability that two students will have birthday this month? (assume that all months have the same number of days to make the numbers simpler). Three students? Four students?
6. You and your friend have a bag of 20 coins that you want to split evenly. If you don't have time to count, what is the probability that you will split it exactly evenly (10+10)? What is the probability that you will make it uneven by 1 coin (11+9 either way)? by 2 coins (12+8 either way)? *Use a calculator to find approximate numbers*