MATH 6: HANDOUT X
PERMUTATIONS AND COMBINATIONS

Permutations with repetitions

If there are identical objects among the selections, then there will be some overcounting that we need to correct. We do that by dividing to the number of arrangements of the repeated objects. Let's take a look at the number of ways to arrange the letters in the word WALL. If the letters would be distinct, then the number of arrangements (permutations) would be $4!$. But because the letter L is repeated, we need to divide by $2!$, the number of ways to arrange 2 objects (both L letters). The answer is $\frac{4!}{2!} = 12$. Another example: how many different arrangements can be formed with the letters from the word ALLELE? The answer is: $\frac{6!}{3!2!} = 20$. We divide by $3!$ because the three letters L are indistinguishable, and $2!$ for the Es.

Cyclic permutations

Imagine we now want to make a dancing circle with n people. In reality, we only care about who is to the right and to the left of each person, as the whole circle can move around and rotate. In how many ways can we make this circle?

To solve this problem, we first think that the circle has definite and static positions. In this case, we would have a total of $n!$ different arrangements for the circle. Now, if we move each person to the position to its right, we will get exactly the same circle (as everyone will have the same neighbors). We can repeat this process until everyone is back to their starting positions, which will require n steps. This means that we counted n arrangements that are equivalent as if they were actually different. We therefore need to correct for this overcounting by dividing by n. Therefore, the total number of ways to do this circle is $n!/n = (n-1)!$.

When Order Does Not Matter - Combinations

Combinations are used when order does not matter. For example when a sports team is selected, or a committee is formed, it does not matter which team member is selected first or second or third. What matters is only the fact that they are selected. We can think of this as arrangements that need to be corrected because of overcounting - essentially, everyone on the selected team is 'identical'. Combinations can be written as ${}_n\mathrm{C}_\mathrm{k}$ for k objects chosen out of n objects, we say "n choose k". Combinations can be also written as $\binom{n}{k}$. The two notations are equivalent.

$${}_n\mathrm{C}_\mathrm{k} = \binom{\mathrm{n}}{\mathrm{k}} = \frac{\mathrm{n}!}{\mathrm{k}!(\mathrm{n}-\mathrm{k})!} = \frac{{}_\mathrm{n}\mathrm{P}_\mathrm{k}}{\mathrm{k}!}$$

An easy way to remember how to find the number of combinations, it is more intuitive to first assume that the order matters. Then, we correct for the overcounting we did by assuming the order did matter by dividing by $k!$ similar to what we did with the permutations with repetitions.

Lastly, remember that $n \cdot (n-1)! = n!$, and in general $n \cdot (n-1) \cdot \ldots \cdot (k+1) \cdot k! = n!$

HOMEWORK

1. (a) List all arrangements of letters in the word WALK.
(b) Copy the same list again, except replace all the letters K with the letter L.
(c) Cross out any repeated arrangements in your second list (that is, if you find two identical arrangements of letters, you may cross one out).
(d) Explain why the number of ways to arrange the letters of WALK is double the number of ways to arrange the letters of WALL.

2. In how many ways can you arrange the letters in the word $THAT$?

3. In how many ways can you arrange the letters from the word TICKTOCK?

4. In how many ways can you select a team of 3 from 8 people?

5. In how many ways can you arrange 5 people around a round table? (A table arrangement is considered the same if each one has the same neighbors to the right and left)

6. I want to make a necklace with six beads - I have one bead each in purple, blue, violet, and mauve, and two red beads. I care about the order in which the colors are arranged - how many ways are there to arrange the beads into an order for the necklace? (Assume that the two red beads are identical.)
(a) First, imagine that your necklace has a distinctive clasp. Therefore, the beginning and the end of the necklace are properly defined.
(b) Now, suppose there is no clasp, so that the necklace has no definite start or end point (think of a cyclic permutation).

7. How many 3-letter words can you form using the letters A, B, C or D if you need to use A at least one time and we are allowed to repeat letters?

8. A family has 4 sons and 3 daughters. In how many ways can they be seated on 7 chairs so that at least 2 boys are next to each other? (Hint: Use complement counting. In what case are there no boys sitting next to each other?)

***9.** How many paths can you form on a grid with 3 rows and 4 columns if you start at the lower left corner and finish at the upper right corner? You can go only on the lines of the grid and you can only go to the right or up.

***10.** Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No.1, and the host, who knows what is behind the doors, opens another door, say No.3, which has a goat. He then says to you "Do you want to change your answer to door No.2?". Is it to your advantage to switch your choice?