MATH 6. HANDOUT 22: INVARIANTS

An invariant is something that does not change...

In this class (category) of problems you are given a set of objects (like some numbers) and then some operations that can be performed on these objects. The question is if a new object can be obtained from the given set of objects. For each problem, think about an invariant (a rule, an expression) that doesn't change when the operations are performed. Does the new object satisfy the invariant rule? If no, then you proved that that the new object can not be obtained from the given set. Problems 1 to 3 are also invariant problems but you are asked to predict the final result.

Here is one basic example: The numbers 1 through 6 are written on the board, and I am allowed to add 1 to two of the numbers at a time. Is it possible for me to repeat this action in such a way that all the numbers will end up the same? To illustrate briefly, let's say I focus on the numbers 1 through 3, and I decide to add 1 to 1 and 2, then add 1 to 1 and 2 again, and then add 1 to 1 and 3 - this will end up with the numbers 4, 4, 4, which are all the same. Can I continue this to make all the numbers the same? The answer is no, because whenever I perform this operation, the sum of the numbers increases by 2, so if the sum of the original numbers is odd, then the sum of the numbers after any step is also odd. The sum of 1 through 6 is 21, which is odd, but the sum of six identical numbers is even, therefore I can't make all the numbers the same.

SAMPLE PROBLEMS

We solved the problems below in class — no need to submit them!

1. A coin change machine changes each coin into 5 others. Is it possible to end up with 26 coins?
   Solution: The remained upon division by 4 is always 1, so it's impossible.
2. There are 16 glasses on the table, one of them upside down. You are allowed to turn over any 4 glasses at a time. Can you get all glasses standing correctly by repeating this operation?
   Solution: The number of upside-down glasses is always even (can you see why? – try different cases!). So it's impossible.
3. 6 trees grow in a row, with a distance of 10 meters between them. One bird sits on each tree. When one bird flies to a different tree, another bird flies in an opposite direction by the same distance. Can all birds get together on one tree?
   Solution: Number the trees from 1 to 6, and gave each bird a number corresponding to a tree it is on. At the initial moment, the sum of birds' numbers is $1 + 2 + 3 + 4 + 5 + 6 = 21$. This sum doesn't change, since if one bird increases its number by x, the other bird will decrease its number by x. For birds to be on one tree, this sum should be even, so it's impossible.

HOMEWORK

1. Numbers 1 through 20 are written on the blackboard. Every minute two of the numbers are erased and replaced by their sum. Can you predict which number will be written on the board at the end?
2. Students have written on the blackboard 2011 "+" signs end 2011 "-" signs. Every minute a pair of signs is erased and replaced by a single "+" if they were equal or a single "-" if they were different. Can you predict which sign will be written on the board at the end? [Hint: look at the product]
3. Numbers 1 through 20 are written on the blackboard. Every minute a pair of numbers a, b are erased and replaced by $a + b - 1$. Can you predict which number will be written on the board at the end?
4. In the alphabet used by the tribe OUO there are only two letters, O and U. Two words in their language are synonyms if one word can be obtained from the other by crossing out or adding anywhere in the word the combinations "OU" and "UUOO". Are the words OUU and OOU synonyms?
5. There are 16 glasses on the table, one of them upside down. You are allowed to turn over any 4 glasses at a time. Can you get all glasses standing correctly by repeating this operation?
6. There are 16 glasses on a table, arranged in a 4x4 grid, the glass in the bottom-left corner upside down. You are allowed to turn over any 2x2 square of glasses at a time. Can you get all the glasses standing correctly except the one in the top-right corner?

7. Is it possible for 17 people to be Snapchat friends with each other in such a way that each person is friends with exactly three other people in the group? [Hint: how many friendships would there be?]

***8.** In the country of RGB, there are 13 red, 15 green and 17 blue chameleons. Whenever two chameleons of different colors meet, both of them change their color to the 3rd one (e.g., if red and green meet, they both turn blue). Do you think it can happen that after some time, all chameleons become the same color?[Hint: give each color a numeric value, say 0, 1, 2]

***9.** A band of four thieves are wandering a city that has a perfect square grid road plan. The thieves always face along the direction of one of the grid lines; at any time, they may walk forward to the next intersection in the direction they are facing, or they may turn 90 degrees to face a different direction. Whenever they perform one of these actions, they plunder a silver coin.
The thieves are initially positioned at the corners of downtown, which is itself the shape of a square, and begin with no silver coins. Is it possible for them to meet up at the same intersection facing the same direction in such a way that they plunder a total of an odd number of silver coins?

***10.** There are 16 glasses on a table, arranged in a 4x4 grid, the glass in the bottom-left corner upside down. You are allowed to turn over any 1x4 row or column of glasses at a time. Can you get all the glasses standing correctly except the one in the top-right corner?