

MATH 8: HANDOUT 18

1. POINTS AND LINES

Axiom 1 (Lines). For any two distinct points A, B , there is a unique line containing these points (this line is usually denoted \overleftrightarrow{AB}).

Axiom 2 (Distance). If points A, B, C are on the same line, and B is between A and C , then $AC = AB + BC$.

Axiom 3 (Angles). If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to 180° .

Axiom 4 (Euclid's Parallel Postulate). Let line l intersect lines m, n and angles $\angle 1, \angle 2$ are as shown in the figure below (in this situation, such a pair of angles is called **alternate interior angles**). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.

Theorem 1. If lines l, m intersect, then they intersect at exactly one point.

Theorem 2 (Parallelism). If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.

Theorem 3 (Vertical Angles). Let M be the intersection point of line segments \overline{AB} and \overline{CD} . Then $m\angle AMC = m\angle BMD$ (such a pair of angles are called **vertical**).

Theorem 4 (Perpendicularity). Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to 90° . Then the three other angles are also equal to 90° . (In this case, we say that lines l, m are **perpendicular** and write $l \perp m$.)

Theorem 5. Let l_1, l_2 be perpendicular to m . Then $l_1 \parallel l_2$. Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

Theorem 6. Given a line l and point P not on l , there exists a unique line m through P which is parallel to l .

Theorem 7. Given a line l and a point P not on l , there exists a unique line m through P which is perpendicular to l .

2. TRIANGLES AND CONGRUENCE

Definition 1 (Triangles). A triangle consists of three points (called the vertices of the triangle) and their three corresponding line segments (the sides of the triangle). A triangle with vertices A, B, C is written $\triangle ABC$, and has sides $\overline{AB}, \overline{BC}$, and \overline{CA} , and interior angles $\angle ABC, \angle BCA, \angle CAB$.

Definition 2 (Altitude, Median, Angle Bisector). In triangle $\triangle ABC$,

- The altitude from A is perpendicular to BC (it exists and is unique by Theorem 7).
- The median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC).
- The angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC , then $m\angle BAE = m\angle EAC$).

Definition 3 (Isosceles Triangles). A triangle is **isosceles** if two of its sides have equal length. The two sides of equal length are called **legs**; the point where the two legs meet is called the **apex** of the triangle; the other two angles are called the **base angles** of the triangle; and the third side is called the **base**.

Definition 4 (Congruence). Congruence between two objects of the same type indicates the following:

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D , then the line segments \overline{AB} and \overline{CD} are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC, \triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}, \overline{CA} \cong \overline{FD}, \angle ABC \cong \angle DEF, \angle BCA \cong \angle EFD,$ and $\angle CAB \cong \angle FDE$.

Definition 5 (Locus). A locus is the set of all points satisfying a certain property.

The concept of a locus is quite versatile; we will use it to describe lines and circles, but other figures can be described as well. For example, given two points A, B , the locus of points P whose sum of distances to A, B is a fixed constant (i.e. $AP + BP = c$ for some c) is called an *ellipse*. But that is a fun fact, and we won't dive into this type of figure in our geometry journey for now.

Axiom 5 (SAS Congruence). *If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Axiom 6 (ASA Congruence). *If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.*

Axiom 7 (SSS Congruence). *If two triangles have three sides congruent, then the triangles are congruent.*

Theorem 8 (Triangle Angle Sum). *The sum of the three interior angles of any triangle is 180° .*

Theorem 9 (Isosceles Base Angles). *If $\triangle ABC$ is isosceles, with base AC , then $m\angle A = m\angle C$. Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC .*

Theorem 10 (Isosceles Median, Altitude, Angle Bisector). *If B is the apex of the isosceles triangle ABC , and BM is the median, then BM is also the altitude, and is also the angle bisector, from B .*

Theorem 11 (Greater Sides Opposite Greater Angles). *In $\triangle ABC$, if $m\angle A > m\angle C$, then we must have $BC > AB$.*

Theorem 12 (Greater Angles Opposite Greater Sides). *In $\triangle ABC$, if $BC > AB$, then we must have $m\angle A > m\angle C$.*

Theorem 13 (The triangle inequality). *In $\triangle ABC$, we have $AB + BC > AC$.*

Theorem 14 (Perpendicular Bisector). *The locus of points equidistant from a pair of points A, B is a line l which is perpendicular to \overline{AB} and goes through the midpoint of AB . This line is called the *perpendicular bisector* of \overline{AB} .*

Theorem 15 (Angle Bisector). *For an angle ABC , the locus of points inside the angle which are equidistant from the two sides BA, BC is the ray \overrightarrow{BD} which is the angle bisector of $\angle ABC$.*

Theorem 16. *In any triangle, the following results hold.*

1. *The perpendicular bisectors of three sides meet at a single point.*
2. *The three angle bisectors meet at a single point.*
3. *The three altitudes meet at a single point.*

Note: In original handouts, one part of this theorem was stated as Theorem 16, and other parts were given as HW problems.

3. QUADRILATERALS

Definition 6 (Quadrilaterals). A figure with four vertices, four corresponding sides, and four enclosed angles is called a *quadrilateral*; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). Additionally, a quadrilateral is called

- a *parallelogram*, if both pairs of opposite sides are parallel.
- a *rhombus*, if all four sides have the same length.
- a *trapezoid*, if one pair of opposite sides are parallel. (these sides are called *bases*) and the other pair is not.
- a *rectangle*, if all its angles are equal (and thus equal to 90°).

Theorem 17 (Parallelogram). *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Theorem 18 (Parallelogram 2).

1. Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.
2. Let $ABCD$ be a quadrilateral such $AB = DC$, and $\overline{AB} \parallel \overline{DC}$. Then $ABCD$ is a parallelogram.

Theorem 19 (Rhombus). Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.

Definition 7 (Midline). A midline of a triangle $\triangle ABC$ is a segment connecting the midpoints of any two sides; a midline of a trapezoid $ABCD$ with bases \overline{AB} , \overline{CD} is the segment connecting the midpoints of the non-bases \overline{AD} , \overline{BC} .

Theorem 20 (Midline of a Triangle). If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{AD} \parallel \overline{AC}$.

4. CIRCLES

Definition 8.

- A circle ω with center at O and radius r is the set of all points P such that $OP = r$.
- A radius is any line segment from O to a point A on ω
- A chord is any line segment between distinct points A , B on ω
- A diameter is a chord that passes through O ,
- A tangent line is a line that intersects the circle exactly once; if the intersection point is A , the tangent is said to be the tangent through A .

Theorem 21 (Tangent line). Let A be a point on circle ω centered at O , and m a line through A . Then m is tangent to ω if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to ω at A .

Theorem 22 (Chord perpendicular bisector). Let \overline{AB} be a chord of circle ω with center O . Then O lies on the perpendicular bisector of \overline{AB} . Moreover, if C is on \overline{AB} , then C bisects \overline{AB} if and only if $\overline{OC} \perp \overline{AB}$.

Theorem 23. Let ω_1 , ω_2 be circles with centers at points O_1 , O_2 that intersect at points A , B . Then $\overline{AB} \perp \overline{O_1O_2}$.

Theorem 24 (Tangent circles). Let ω_1 , ω_2 be circles that are both tangent to line m at point A . Then these two circles have only one common point, A . Such circles are called tangent.

Theorem 25 (Inscribed angle). Let A , B , C be on circle λ with center O . Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ .

5. CONSTRUCTION

Definition 9. Euclidean is a straightedge-compass construction game app, available on Google Play and the App Store; you can read about it at <https://www.euclidean.xyz> and https://twitter.com/euclidean_app

Definition 10. A straightedge is a tool that can construct a line through any two points; a compass is a tool that can construct a circle centered at a given point with radius congruent to a given line segment. A construction is a process whereby one constructs a geometric object by using tools to create other objects and then constructing any desired intersection points of the objects; a straightedge-compass construction is a construction using only straightedge and compass as tools.

In general, to provide a straightedge-compass construction of a figure, you may use any previous straightedge-compass constructions you have made as tools - for example, you may construct a parallel to any line through any point, since we have proven that a straightedge-compass construction of this is possible for any line and point.

So far, we have constructed the following items with straightedge and compass, which you may use in your constructions:

- A line segment \overline{CD} congruent to \overline{AB} given the points A , B , and C ;
- An angle at a given ray congruent to another given angle;
- A line parallel to a line l and through a point P ;

- A line perpendicular to a line l and through a point P ;
- The perpendicular bisector of a line segment \overline{AB} ;
- The center of a circle;
- The medians, altitudes, and angle bisectors of a triangle;
- A circle that passes through three given points.

6. HOMEWORK

1. Try to get through as many levels of Euclidean as possible; the goal is to do the first 5 levels (up to Epsilon).
2. Prove that a diameter is the longest chord of a circle (i.e., any chord that is not a diameter has smaller length than a diameter).
3. (a) Prove that a line cannot intersect a circle at three distinct points.
(b) Prove that two circles cannot intersect at three distinct points.
4. Suppose circles λ, ω with centers O, P respectively intersect at point A such that $m\angle OAP = x^\circ$ for some $x < 180$.
(a) Prove that the tangents to λ, ω at A also intersect with an angle of measure x° .
(b) Let B be the other point of intersection of λ, ω ; prove that $\angle OBP \cong \angle OAP$.