MATH 8: HANDOUT 16

Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

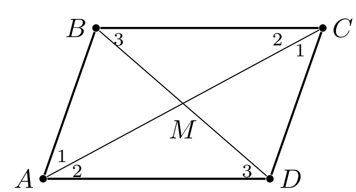
Definition 1. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 17. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*



Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$.

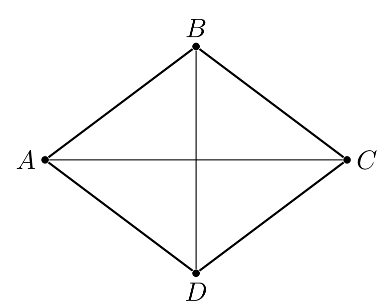
Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. $\square$

Theorem 18.

1. *Let $ABCD$ be a quadrilateral such that opposite sides are equal: $AB = DC$, $AD = BC$. Then $ABCD$ is a parallelogram.*
2. *Let $ABCD$ be a quadrilateral such $AB = DC$, and $\overline{AB} \parallel \overline{DC}$. Then $ABCD$ is a parallelogram.*

Proof of the first part is left to you as an exercise (see homework problem 3); proof of the second part is similar.

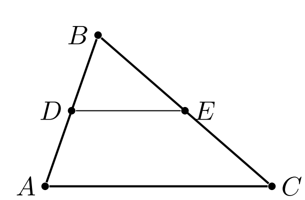
Theorem 19. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*



Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem **??** that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 10 in Assignment 14, it is also the altitude. $\square$

Midline of a triangle and trapezoid

Definition 2. A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.



Theorem 20. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

The proof of this theorem is also given as a homework; it is not very easy.

Constructions with ruler and compass

Now that we know when two geometric objects are the same (via congruence), it makes sense to ask if we can produce figures with specific properties of interest — for example, if we can reproduce a given angle somewhere else so that the resulting angle is congruent to the original. Traditionally, such constructions are done using straight-edge and compass: straight-edge tool which can construct lines, and the compass tool which can construct circles. More precisely, it means that we allow the following basic operations:

- Draw (construct) a line through two given or previously constructed distinct points. (Recall that by axiom 1, such a line is unique).
- Draw (construct) a circle with center at previously constructed point O and with radius equal to distance between two previously constructed points B, C
- Construct the intersections point(s) of two previously constructed lines, circles, or a circle and a line.

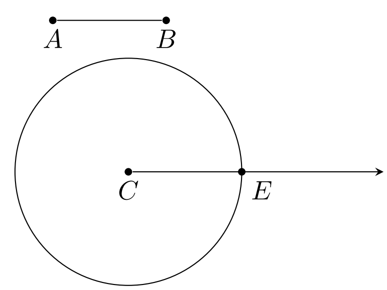
All other constructions (e.g., draw a line parallel to a given one) must be done using these elementary constructions only!!

Constructions of this form have been famous since mathematics in ancient Greece.

Here are some examples of constructions:

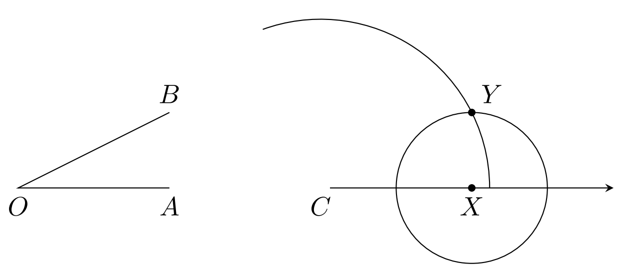
Example 1. Given any line segment $\overline{AB}$ and ray $\overrightarrow{CD}$, one can construct a point E on $\overrightarrow{CD}$ such that $\overline{CE} \cong \overline{AB}$.

Construction. Construct a circle centered at C with radius AB. Then this circle will intersect $\overrightarrow{CD}$ at the desired point E. □

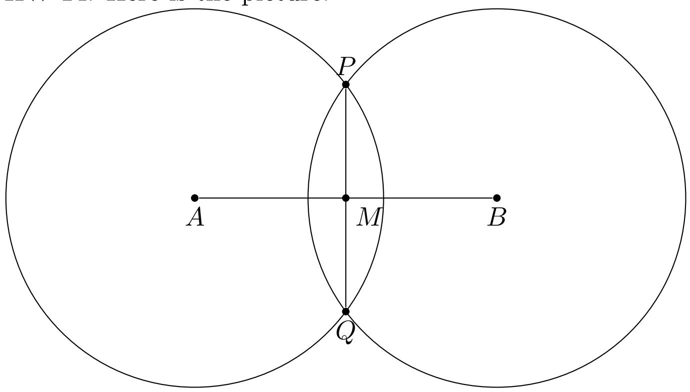


Example 2. Given angle $\angle AOB$ and ray $\overrightarrow{CD}$, one can construct an angle around $\overrightarrow{CD}$ that is congruent to $\angle AOB$.

Construction. First construct point X on $\overrightarrow{CD}$ such that $CX \cong OA$. Then, construct a circle of radius OB centered at C and a circle of radius AB centered at X. Let Y be the intersection of these circles; then $\triangle XCY \cong \triangle AOB$ by SSS and hence $\angle XCY \cong \angle AOB$. □



Example 3. Given a segment AB, one can construct the perpendicular bisector of AB; this was discussed in HW 14. Here is the picture:



In particular, this allows one to construct the midpoint of a segment.

Homework

1. Let $ABCD$ be a rectangle (i.e., all angles have measure 90°). Show that then, opposite sides are equal.

2. (a) Prove that in a rectangle, diagonals are equal length.
(b) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.

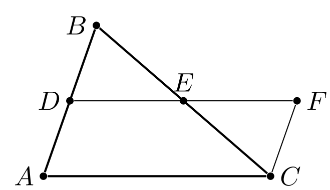
3. Prove part 1 of Theorem **??**.

4. Prove that if in a quadrialteral $ABCD$ we have $AD = BC$, and $\overline{AD} \parallel \overline{BC}$, then $ABCD$ is a parallelogram.

5. Prove Theorem **??** by completing the steps below.

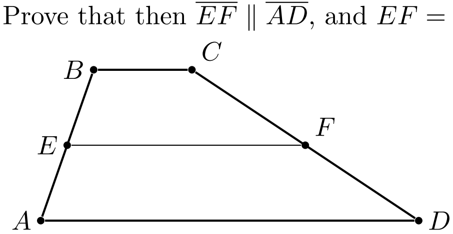
Continue line DE and mark on it point F such that $DE = EF$.

(a) Prove that $\triangle DEB \cong \triangle FEC$
(b) Prove that $ADFC$ is a parallelogram (hint: use alternate interior angles!)
(c) Prove that $DE = \frac{1}{2}AC$

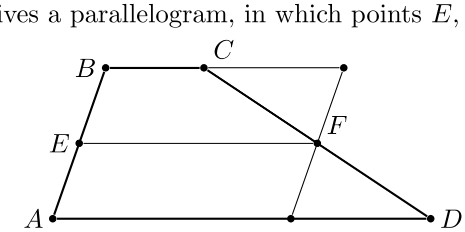


6. Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively.

Prove that then $\overline{EF} \parallel \overline{AD}$, and $EF = (AD + BC)/2$.



[Hint: draw through point F a line parallel to AB, as shown in the figure below. Prove that this gives a parallelogram, in which points E, F are midpoints of opposite sides.]



7. Do the following straightedge and compass constructions. In all cases, you have to not only describe the construction, but also prove why it gives the correct answer.

(a) Given a line l and a point P not on l, construct the perpendicular to l through P (hint: can you construct an isosceles triangle with base on line l and apex at P?)

(b) Given an angle $\angle ABC$, construct the angle bisector. [Hint: again, construct an isosceles triangle!]

(c) Given a circle, find its center. [Hint: recall that the perpendicular bisector is the locus of points equidistant from two given points.]