CHAPTER 10

PASCAL'S TRIANGLE APPLICATIONS

10.1 Pascal Triangle and Combinations

Combination: the choice of k things from a set of n things without replacement and where order does not matter.

Pascal's Triangle Construction 6x6

Consider a grid that has 6 rows of 6 squares in each row with the lower left corner named A and upper right corner named B. Suppose that from the starting point A you can go one step up or one step to the right at each move. This is continued until the point B is reached. How many different paths from A to B are possible?

Let us recall from last time: If we start at A and move towards B, we find we can follow the path RRRRRUUUUU (where R = Right one unit, U = Up one unit). Since we can have an R in any place of our 10 moves, and we only have two choices for moves: right and up, the number of how many good routes we have can be found by finding how many combinations of 5 R's we can have in our 10 moves.

$$\mathbf{C}(10,5) = \mathbf{C}_5^{10} = {}_{10}C_5 = \binom{10}{5}\text{ , where } {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Algebraic properties of combinations

Pascal's Triangle is symmetric

In terms of the binomial coefficients, ${}_nC_k =_n C_{n-k}$.

Algebraic proof using the formula for the combinations/binomial coefficient

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Graphical proof using the construction of the triangle, the ${}_nC_k$= the number of ways to get from the bottom corner to a given (n,k) on the chess board by only moving up snd to the right. For each path to (n,k) there is a path to (k,n) (its symmetrical with respect to the diagonal of the (n,n) chess table

10.2 Binomial Formula

It is possible to expand the power $(x+y)^n$ into a sum involving terms of the form ${}_nC_k \cdot x^k y^{n-k}$, where the exponents k and (n-k) are nonnegative integers, and the coefficient ${}_nC_k$ represents the number of possible ways of choosing k things from a set of n things without replacement and where order does not matter.

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

.

$(x+y)^n$	Pascal's Triangle Line
n=1	1,1
n=2	1,2,1
n=3	1,3,3,1
n= 4	1,4,6,4,1
n= 5	1,5,10,5,1

Interpretation of the formula

If we write $(x+y)^n$ as a product

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

then, according to the distributive law of multiplication, there will be one term in the result for each choice of either "x" or "y" from each of the $(x+y)$ present in the product.

For example for n=3, $(x+y)^n = (x+y)(x+y)(x+y)$ there will only be one term x^3 corresponding to choosing "x" from each $(x+y)$. However, there will be several terms of the form x^2y one for each way of choosing exactly two binomials $(x+y)$ from the product $(x+y)(x+y)(x+y)$ to contribute a "x". How many ? Exactly combinations of n choosing 2.

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx + xxy + xyx + \underline{xyy} + yxx + \underline{yxy} + \underline{yyx} + yyy\\ &= x^3 + 3x^2y + \underline{3xy^2} + y^3.\end{aligned}$$

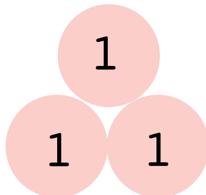
Why is it similar to the Pascal's Triangle ? An intuitive one is to associate to an "x" an "U" up move on the chessboard. Thus, each way of arriving to x^2y by choosing exactly two binomials $(x+y)$ to contribute a "x" has a sequence of "U"s and "R"s : xxy=UUR, xyx=URU, yxx=RUU. Notice that the order does not matter, because multiplication is commutative.

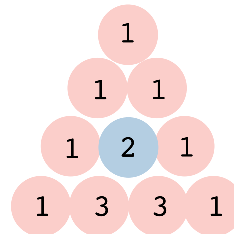
10.3 Problems

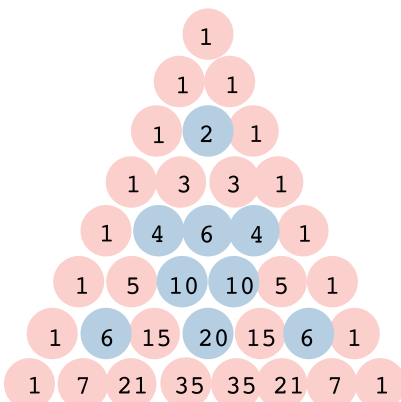
Please do not compute the numerical results, instead try to write them using the factorials!

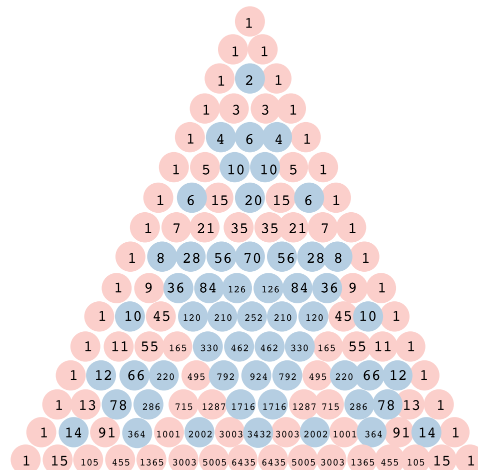
1. A school has 3800 girls and 1200 boys. In how many ways can they create a 6 people student council? In how many ways can they do it if the number of girls and number of boys should be equal? In how many ways can they form a team of 11 for girls volleyball?

2. How many different 7-letter ways of arrange the letters in the word "student" are there? How many different 5-letter ways of arrange the letters in the word "student" are there?









3. The county-mandated guidelines at a certain community college specify that for the introductory English class, the professor may choose one of three specified novels, and choose two from a list of 5 specified plays. Thus, the reading list for this introductory class is guaranteed to have one novel and two plays. How many different reading lists could a professor create within these parameters?

4. There are 10 horses in a horse show competition and the first three horses receive a prize. How many possible prize winning orders are there ?

5. Expand using Pascal's Triangle $(x+y)^8$ and $(x+y)^{10}$.

6. Expand using Pascal's Triangle $(1+x)^5$, $(x+2y)^4$, $(2x-1)^5$ and $(x+\frac{1}{x})^4$

7. What is the coefficient of the term x^3 in $(x+1)^6$?

8. What is the coefficient of the term x^8 in $(x^2+1)^4$?

9.* What are the coefficients of the term $x^1 2$ and respectively x^9 in $(x^2-2x)^6$?

10.* Which of the numbers in Pascal triangle are even? Color the odd and even numbers differently, and compare the triangles that end at the powers of 2. Can you guess the pattern, and then carefully explain why it works?