

**MATH 10**  
**ASSIGNMENT 7: SYSTEMS OF LINEAR EQUATIONS**  
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SYSTEMS OF LINEAR EQUATIONS

A number of practical problems give a number of linear equations which have to be simultaneously solved. For example, suppose students A and B went to the supermarket. Student A bought a bag of cookies and two cans of soda, paying five dollars. Student B bought two bags of cookies and a can of soda and paid seven dollars. How much does each item cost?

It is easy to see that this problem can be phrased in terms of a *system of linear equations*. Let  $c$  denote the price of the bag of cookies and  $s$  the price of a can of soda. Then

$$\begin{aligned}c + 2s &= 5 \\ 2c + s &= 7\end{aligned}$$

An intuitive method to solve this problem is that of substitution. First we solve the first equation for  $c$ :

$$(1) \quad c = 5 - 2s.$$

Then we substitute this result in the first equation and solve for  $s$ :

$$2(5 - 2s) + s = 7 \Rightarrow s = 1$$

Finally, we go backwards, substituting  $s = 1$  back in equation (1):  $c = 5 - 2(1) = 3$ .

We will now start the study of general systems of linear equations, with  $m$  equations and  $n$  variables. They will look something like this:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}$$

To such a system of equations we associate the *matrix of coefficients*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

and the *vector of constants*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We can combine these two into the *augmented matrix*,

$$A|\mathbf{b} = \left[ \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right],$$

For example, if the systems of linear equations is

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 2 \\ 4x_1 - 7x_2 + 5x_3 &= 1,\end{aligned}$$

then we get

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -7 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad A|\mathbf{b} = \begin{bmatrix} 2 & 1 & 3 & | & 2 \\ 4 & -7 & 5 & | & 1 \end{bmatrix}.$$

#### ELEMENTARY ROW OPERATIONS

The main idea of the method we will use to solve an arbitrary system of linear equations is to transform it to a simpler form. The transformation should not change the set of solutions. To do this, we will use the following elementary operations:

- Exchange two equations (= two rows of the augmented matrix)
- Multiply both sides of an equation (= one row of augmented matrix) by a non-zero number
- Add to one equation a multiple of another (= add to one row of the augmented matrix a multiple of another)

Applying these operations to bring your matrix to a simpler form is called *row reduction*, or *Gaussian elimination*

#### SIMPLE EXAMPLE

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 5 \\ x_1 - x_2 &= -1 \\ -x_1 + x_2 + x_3 &= 5 \end{aligned}$$

The augmented matrix is

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 1 & 5 \end{array} \right]$$

Using row operations, we can bring it to the form

$$\left[ \begin{array}{ccc|c} 1 & -2 & 2 & 5 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

so the solution is

$$\begin{aligned} x_3 &= 4 \\ x_2 &= -6 + 2x_3 = 2 \\ x_1 &= 5 - 2x_3 + 2x_2 = 1 \end{aligned}$$

#### ROW ECHELON FORM

In general, using row operations, every system can be brought to a form where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$\left[ \begin{array}{cccccc|c} X & * & * & * & * & * & * \\ 0 & 0 & 0 & X & * & * & * \\ 0 & 0 & 0 & 0 & X & * & * \end{array} \right]$$

(here  $X$ 's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with  $X$ 's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chosen arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$\begin{aligned}x_1 + x_2 + x_3 &= 5 \\x_2 + 3x_3 &= 6\end{aligned}$$

variables  $x_1, x_2$  are pivot, and variable  $x_3$  is free, so we can solve it by letting  $x_3 = t$ , and then

$$\begin{aligned}x_2 &= 6 - 3x_3 = 6 - 3t \\x_1 &= 5 - x_2 - x_3 = -1 + 2t\end{aligned}$$

Make sure you understand what the meaning here is: for any real number  $t$ ,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 2t \\ 6 - 3t \\ t \end{bmatrix}$$

is a solution.

#### HOMEWORK

1. Solve the following system of equations

$$\begin{aligned}w + x + y + z &= 6 \\w + y + z &= 4 \\w + y &= 2\end{aligned}$$

2. Solve the system of equations with the following matrix

$$\left[ \begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{array} \right]$$

3. Solve the following system of equations

$$\begin{aligned}x_1 + x_2 + 3x_3 &= 3 \\-x_1 + x_2 + x_3 &= -1 \\2x_1 + 3x_2 + 8x_3 &= 4\end{aligned}$$

4. Consider the system of equations

$$\begin{aligned}3x - y + 2z &= b_1 \\2x + y + z &= b_2 \\x - 7y + 2z &= b_3\end{aligned}$$

- (a) If  $b_1 = b_2 = b_3 = 0$ , find all solutions
  - (b) For which triples  $b_1, b_2, b_3$  does it have a solution?
5. Consider a system of 4 equations in 5 variables.
    - (a) Show that if the right-hand side is zero, then this system must have a non-zero solution.
    - (b) Is it true if the right-hand side is non-zero?