COMMUNICATING VESSELS AND ARCHIMEDES FORCE.

MAY 9, 2021

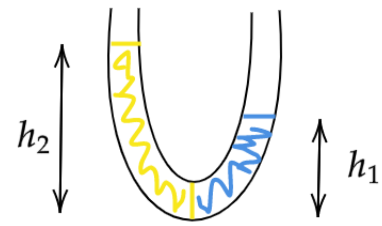
Theory recap

Last class recap. Last time we started discussing physics of liquids and learned how pressure grows with depth:

$$p(h) = p_0 + \rho g h.$$

Today we will look at two consequences of this formula: communicating vessels and the Archimedes force.

Communicating vessels. Imagine a U-shaped vessel. Let us fill it with two liquids which do not mix with each other, for example water and vegetable oil. Water will be poured into the right tube and oil - into the left tube. Imagine that the amount of liquids poured is exactly such that the interface point will be at the very bottom. Then how would the levels of these two liquids be related?



The main principle we should use is that pressure at the interface point created by both liquids should be the same. It is a necessary condition for the liquid to be in equilibrium. If water level is h_1 and oil level is h_2, we could calculate the pressure at the level of the bottom of the vessel in two ways. On one hand, it is created by a column of water of height h_1:

$$p = p_0 + \rho_1 g h_1,$$

where p_0 is the atmospheric pressure and ρ_1 is water density. On the other hand, the same pressure is created by column of oil (with density ρ_2) of height h_2:

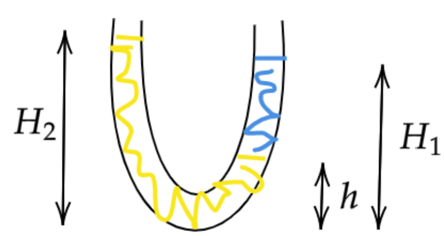
$$p = p_0 + \rho_2 g h_2.$$

By setting the two expressions for p equal, we find

$$p_0 + \rho_1 g h_1 = p_0 + \rho_2 g h_2 \implies \rho_1 g h_1 = \rho_2 g h_2 \Longrightarrow \rho_1 h_1 = \rho_2 h_2. \tag{1}$$

We see that a more dense liquid has a lower level. For instance, water has higher density than oil: $\rho_1 = 1000$ kg/m^3 vs. $\rho_2 = 900$ kg/m^3 and therefore water level will be lower than oil level.

Let us also consider a case where interface between the two liquids is not at the bottom of the vessel. For instance, let water be present both in the right and left tube and water be poured above oil in the right tube. Dimensions are shown on the figure: oil level in the left tube is H_2, oil level in the right tube - h, liquid level in the right tube measured from the bottom is H_1 which gives water level $H_1 - h$ (total liquid level minus oil level).



The basic principle i the same: we should set equal pressure created by oil and water at the interface point. On one hand, water has pressure

$$p = p_0 + \rho_1 g(H_1 - h)$$

because its level above the interface is $H_1 - h$. On the other hand, oil level above the interface is $H_2 - h$. We could understand it if we start going down from the top of left tube: first we go H_2 down to reach the bottom and the pressure rises along the way. But then we go up the right tube up to height h and pressure decreases to the value

$$p = p_0 + \rho_2 g(H_2 - h).$$

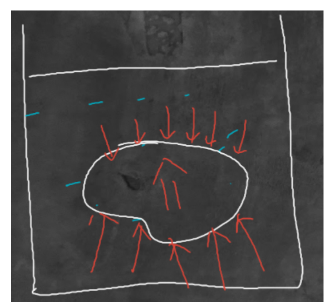
Again, set two expressions for pressure equal, cancel the common term of atmospheric pressure and obtain

$$\rho_1 g(H_1 - h) = \rho_2 g(H_2 - h) \Longrightarrow \rho_1(H_1 - h) = \rho_2(H_2 - h)$$

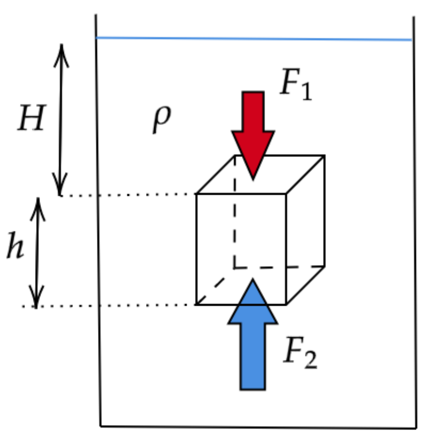
This formula has the same meaning as equation (1) we obtained for interface being at the bottom. In both cases product of height measured from the interface level and density must be the same for both liquids.

Archimedes force. When an object is submerged underwater there is a buoyant force trying to push it up. For some objects this force is enough to make it float on the surface, like for boats, ships or humans. For other it is less noticeable. This buoyant force is also known as Archimedes force and it is directly related to properties of pressure in liquids which we discussed.

The origin of this force is very simple. Imagine an submerged object. Pressure grows with depth, so pressure at the level of the bottom of the object is higher, then at the level of its' top (as shown on the figure). Water pressure at the top of the object results in a force pushing the object down, while pressure at the bottom results in a force pushing the object up. Because pressure is higher at the bottom level, force pushing up is stronger and there is a net force pushing up.



There is also a really simple expression for the Archimedes force. Today we will derive it for a simple shape. Imagine our object to be a rectangular parallelepiped with height h and cross section area A. For such a shape expressions for vertical forces are simple and we can calculate them. As we already said, water pushes the top side down with some force which we denote as F_1. If the parallelepiped's top side is submerged to depth H, pressure at the level of top side is



$$p_1 = p_0 + \rho g H$$

where as usual p_0 is atmospheric pressure, ρ is water density. Force is pressure times area, therefore

$$F_1 = p_1 A.$$

Similarly, pressure at the level of bottom side of the parallelepiped is

$$p_2 = p_0 + \rho g(H + h)$$

and the force acting on this side up is

$$F_2 = p_2 A.$$

The net vertical force applied by the water on our parallelepiped is the difference between F_2 and F_1:

$$F_A = F_2 - F_1 = (p_2 - p_1)A = \rho g h A = \rho g V.$$

At the last expression we replaced product of height and area by volume of the parallelepiped V. THe result is very simple: it does not depend on atmospheric pressure or depth at which the object is submerged. This is understandable, because atmosphere or depth provide some constant pressure which will cancel when we calculate pressure difference between the top and the bottom. Next time we will discuss that the formula

$$F_A = \rho g V$$

which gives Archimedes force as a product of liquid density, free fall acceleration and object's volume is actually very general and not restricted to a simple parallelepiped case.

Homework

1. Water is in the right tube of a U-shaped tube and dishwashing liquid with density about 1100 kg/m^3 is in the left tube. Interface between the liquids is at the very bottom. What is the level of dishwashing liquid if water level is 1 cm higher than it?

2. Find Archimedes force acting on a solid cube with side 5 cm when it is completely submerged into liquid mercury. Mercury density is 13600 kg/m^3. Suppose the cube is made out of iron (iron density is 7800 kg/m^3). Compare Archimedes force to gravity force acting on the cube. Try to understand what does this relation tell you about the behavior of iron cube submerged in the mercury: will it float or sink?

***3.** Imagine a U-shaped vessel with just water in both tubes. Water level is of course the same in both tubes. Assume the distance between water level and top edge of the tubes (which are at the same level) is 10 cm. What volume of oil could be carefully (assume no mixing) poured in the left tube before one of the liquids overflows the tube? Which of the liquids will overflow first? Cross section area of the vessel is the same everywhere and equal to 3 cm^2. Take density of oil to be 900 kg/m^3.