KINETIC ENERGY

JANUARY 10, 2021

Theory recap

Before the holidays we discussed momentum and momentum conservation law. We learned that for an isolated system of bodies (with no external forces acting on it) total momentum does not change with time. There is also another conservation law you might have heard about: energy conservation law. Energy is a bit trickier than momentum because it has many forms and conservation only applies to the total energy, not individual forms - so, energy could change forms. Some of these forms are kinetic, potential and thermal energy, which will appear in our course. Today we discuss the first of them: kinetic energy.

As we discussed before, an object of mass m moving with velocity v has momentum $p = mv$. Another quantity which is associated with any moving object is kinetic energy. Kinetic energy is equal to the mass of the object times its' speed squared and divided by 2:

$$E_{kinetic} = \frac{mv^2}{2}$$

It might be a bit hard to grasp the importance of kinetic energy without referring to transfer into other kinds of energy. We will do it in detail in the next few classes. For now, we could just qualitatively observe, that an object going upwards with a bigger velocity will go higher. In energy language that would mean that larger kinetic energy got transferred into larger potential energy.

Kinetic energy is a scalar quantity: it is just a number and it does not have a direction. So only speed matters to it, not velocity. A ball moving to the right or to the left with the same speed will have the same kinetic energy (but not the same momentum).

Units of energy can be deduced from the formula for kinetic energy:

$$J = kg \cdot \frac{m^2}{s^2}$$

There is a special name for the unit of energy: joule (after English physicist James Prescott Joule).

For example, consider an 80 kg cyclist moving with speed 5 m/s. What is his kinetic energy? $\dfrac{80\ kg \cdot (5\ m/s)^2}{2} = 1000\ J = 1\ kJ$. Similarly to kilograms and kilometers 1000 joules is called a kilojoule, kJ.

If the speed of the cyclist was doubled, his kinetic energy would become four times larger, because kinetic energy contains square of the speed. For kinetic energy speed is more important than mass, to contrast with momentum for which mass and speed are equally important.

If two cars of different mass have the same momentum, which one has larger kinetic energy: a heavier car or a lighter car? From equality of momenta we have $m_1v_1 = m_2v_2 \Longrightarrow v_2 = \dfrac{m_1v_1}{m_2}$, so for kinetic energy of the 2nd car:

$$\frac{m_2v_2^2}{2} = \frac{m_2\left(\dfrac{m_1v_1}{m_2}\right)^2}{2} = \frac{m_1^2v_1^2}{2m_2} = \frac{m_1v_1^2}{2} \cdot \frac{m_1}{m_2} \Longrightarrow E_{kinetic2} = E_{kinetic1} \cdot \frac{m_1}{m_2}$$

Therefore, a lighter car has bigger kinetic energy because it has higher speed and speed is more important to kinetic energy than mass.

There is another form of the expression for kinetic energy, which may be useful sometimes. Recall that $p = mv$ is the momentum, therefore $v = \frac{p}{m}$ and we could express kinetic energy via momentum and mass, without speed:

$$E_{kinetic} = \frac{mv^2}{2} = \frac{m\left(\frac{p}{m}\right)^2}{2} = \frac{p^2}{2m}$$

From this formula it is very easy to confirm our finding above, that for the same momenta an object with smaller mass has larger kinetic energy.

As was mentioned in the beginning, energy is very important because it is conserved. Even though in many situations energy is transferred between different forms, there are situations in which only kinetic energy is important and it is separately conserved.

Let us look at a collision of two objects. During a collision, some of the initial kinetic energy of these objects can be lost, or dissipated. Many forms of dissipation can occur in a collision: heat and sound can be produced, there could be some structural damage that costs some energy of deformation. It is important that overall energy even in this case is conserved - kinetic energy gets transferred to other forms.

Completely elastic collisions are such in which we can neglect dissipation, so that kinetic energy is conserved. For example, if you throw a rubber ball into a wall, it bounces elastically if its speed after hitting the wall is exactly the same as it was before the collision. Since speed is the same as before, kinetic energy is also the same.

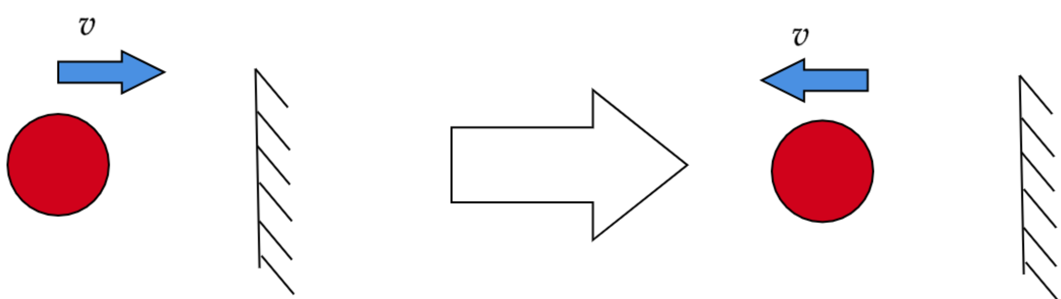

FIGURE 1. Completely elastic collision of a ball with a wall

For one ball that is it, but the situation gets a lot more interesting if we have two balls. For example, this happens in the game of billiard, where one needs to hit some balls on the table with another ball. So let us find out, what happens if balls collide elastically. Let us assume that two balls are of the same mass m and move along a straight line without friction. For simplicity, consider that before collision the first ball moves with velocity v to the right and the second ball is at rest. Let us call their respective velocities after the collision v_1 and v_2, with positive values corresponding to the direction to the right.

Even before saying that the collision is completely elastic we could use momentum conservation law. The total momentum of the two balls is conserved because there are no external horizontal forces acting on them - there is no friction. Initial momentum of the first ball

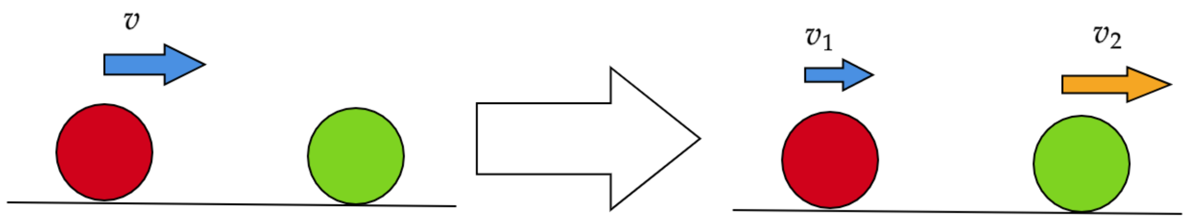

FIGURE 2. Completely elastic collision of two balls

is mv and initial momentum of the second ball is 0, so the total initial momentum is mv. Final momentum of the first ball is mv_1 and final momentum of the second ball is mv_2, so the total final momentum is $mv_1 + mv_2$. Momentum conservation law thus tells us that

$$mv = mv_1 + mv_2 \Longrightarrow \boxed{v = v_1 + v_2}$$

Now let us use the fact that the collision is completely elastic. This means that total kinetic energy before the collision is equal to the total kinetic energy after the collision. Before the collision only one ball moves, so the total kinetic energy is $\frac{mv^2}{2} + 0 = \frac{mv^2}{2}$. After the collision kinetic energy of the first ball is $\frac{mv_1^2}{2}$ and kinetic energy of the second ball is $\frac{mv_2^2}{2}$, so the total kinetic energy is $\frac{mv_1^2}{2} + \frac{mv_2^2}{2}$. Energy conservation law is:

$$\frac{mv^2}{2} = \frac{mv_1^2}{2} + \frac{mv_2^2}{2} \Longrightarrow \boxed{v^2 = v_1^2 + v_2^2}$$

Now we have a system of two boxed equations for the two unknowns (v_1 and v_2) and to solve them just requires maths. From the first equation we could express v_2: $v_2 = v - v_1$ and plug it into the second equation and do a series of simplifications:

$$v_1^2 + (v - v_1)^2 = v^2$$
$$v_1^2 + v^2 - 2vv_1 + v_1^2 = v^2$$
$$2v_1^2 - 2v_1 v = 2v_1(v - v_1) = 0 \Rightarrow \begin{cases} v_1 = 0 \text{ or} \\ v_1 = v \end{cases}$$

Recall that $v_2 = v - v_1$, so the full solutions are $v_1 = 0, v_2 = v$ and $v_1 = v, v_2 = 0$. We have obtained two answers, which one should we pick? Let us remember that the first ball is always to the left from the second ball and both v_1 and v_2 are directed to the right. Therefore it could not be that $v_1 > v_2$, because the first ball should fall behind the second one. So we have to choose $\boxed{v_1 = 0, v_2 = v}$.

This actually means something nice: as a result of the collision the balls just exchanged their velocities. The first ball comes to a stop while the second gets all of its speed. This is close to what one actually observes in billiard (which means that collisions in billiard are almost completely elastic). The result that after an elastic collision balls will exchange their velocities actually would hold for any initial velocities of the two balls of the same mass (if you want to know how to prove it, the easiest way would be to go to the reference frame in which one of the balls is at rest. Try to finish the proof by yourself, if you want).

But you may ask, why did we get two solutions? What does the discarded solution correspond to? We could observe that the discarded solution $v_1 = v, v_2 = 0$ corresponds to the situation before the collision: the first ball moves and the second is at rest. This is indeed a solution of momentum conservation law and energy conservation law, because energy and momentum are of course conserved if velocities just stay what they are. But as we discussed, velocities can't stay the same after the collision happened, so we discard this solution.

Remember how I said that conservation laws help a lot? This is the first example of what could momentum conservation and energy conservation do together. All we needed to know are the initial velocities of the balls, and then it is quite easy to find what their final velocities will be.

Homework

1. Calculate kinetic energy of a falling stone with a mass of 10 kg after 3 seconds of falling.

2. Imagine that both the mass and the speed of a moving object increased 2 times. How did its kinetic energy change?

3. A runner moves with speed $v = 4\ m/s$ and has momentum $p = 250\ kg \cdot m/s$. Find kinetic energy of the runner. Derive a general formula for kinetic energy in terms of v and p.

***4.** 2020 identical balls are at rest, placed on a straight line at the interval of 1 m between the neighboring balls (see the figure below). 2021-st ball (the same as others) comes from the left with speed 1 m/s. How much time will pass between the first collision in this system and the last collision? How will all the balls move after the last collision? Size of the balls is much smaller than the distance between them.

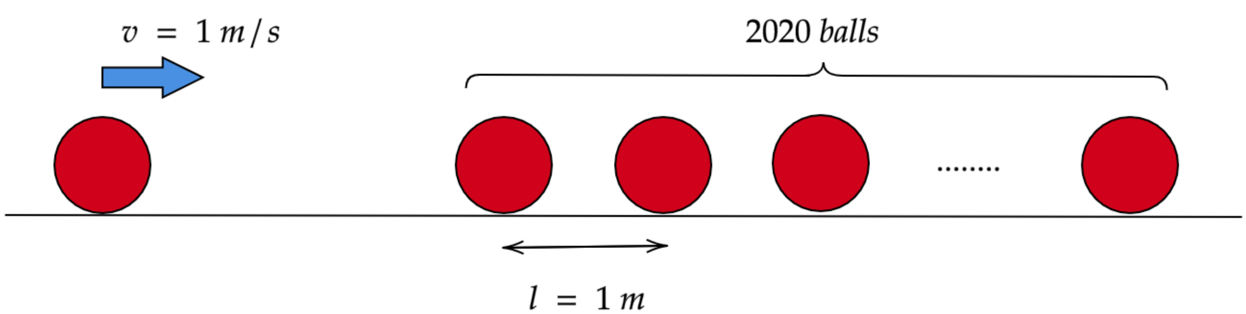

Figure 3. To problem *4