MOMENTUM AND MOMENTUM CONSERVATION
DECEMBER 13, 2020

Theory recap

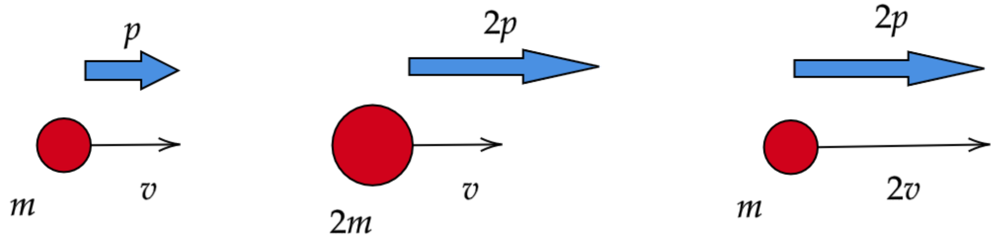
Imagine that a car and a big truck are moving with the same speed. Which one of them is harder to stop? One way to answer this question is from the point of view of Newton's second law. The truck has larger mass, so to supply it with the same acceleration as the car, we would need to apply larger force.

But there is another reasoning which is related to Newton's second law but formulated differently. We introduce a new quantity called momentum which is equal to product of mass and velocity:

$$\vec{p} = m\vec{v}$$

We see that for larger mass and for larger speed momentum gets larger. A truck has larger momentum than a car, so it is harder to stop the truck.

As we see from the equation defining the momentum, it is a vector. It has magnitude equal to product of mass and speed and its direction is the same as direction of velocity. This is illustrated on a figure below.



Units of momentum are the product of units of mass and speed: $kg \cdot \dfrac{m}{s} = \dfrac{kg \cdot m}{s}$. Unlike force, there is no special name for this unit.

Looking at the definition of momentum we see that there are two ways to change it: either to change mass or to change velocity. We will not consider changing mass. Change in velocity requires acceleration. By Newton's second law acceleration requires a force. So, change in momentum requires some force. Note that we do not say anything new here, it is Newton's first law expressed in different terms. If no net force is applied to a body, its momentum stays the same.

How momentum changes when there is a force will be discussed next time. Today we will look at momentum of a system of bodies instead. Say we have two blocks, one with mass m_1 and velocity $\vec{v}_1$ and the other with mass m_2 and velocity $\vec{v}_2$. We could calculate their total momentum by adding their individual momenta:

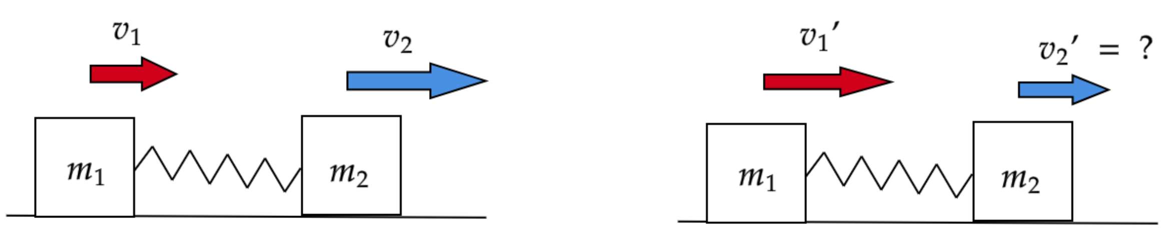
$$\vec{p}_{tot} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2$$

Remember that for one body if there is no net force, its momentum stays constant. There is a generalization of this to our system: if there is no net external force, the total momentum of a system stays constant.

First, let us understand what external force means. We have two bodies in our system. They interact with each other and with some external objects (the Earth, floor, etc.). Forces of interaction between the objects in the system are called internal. They *always* come in pairs of equal and opposite forces: if m_1 acts on m_2 with force $\vec{F}$ then by Newton's third law m_2 acts on m_1 with force $-\vec{F}$. Therefore, when we consider the whole system and add up all the forces, these internal forces *always* cancel out. On the other hand, forces between the objects in the system and external objects, called external forces, do not cancel out. So only external forces could change properties of the whole system, such as total momentum. And if there is no net external force, momentum does not change.

The statement that something does not change (stays constant in time, is conserved) is called a conservation law. We have formulated momentum conservation law: total momentum is conserved if there is no net external force. It is a very-very important law. There are many cases when using it is much simpler than using Newton's laws directly. Moreover, though it will not arise in our class, momentum conservation law still stays applicable even when Newton's laws fail - like in relativity theory and to some extent in quantum theory. This could give you an appreciation of its importance.

Let us apply momentum conservation law to a case when we have two blocks connected by a spring on a horizontal table without friction. Suppose the blocks have masses m_1 and m_2 and at some moment of time they respectively have velocities v_1 and v_2 directed to the right. Then after moving for some time block m_1 has velocity v_1' directed to the right. Let us ask what is velocity of the block m_2 at the same time.



The main idea is that since there is no friction, there is no net external force (because gravity is balanced by the normal force). Therefore total momentum is conserved. Let us write total momentum at the first moment choosing positive direction to the right:

$$p_{tot} = p_1 + p_2 = m_1 v_1 + m_2 v_2$$

At the second moment total momentum is expressed via new velocities:

$$p_{tot} = p_1' + p_2' = m_1 v_1' + m_2 v_2'$$

By setting one p_{tot} equal to the other we get

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2 \Longrightarrow m_2 v_2' = m_1 v_1 + m_2 v_2 - m_1 v_1' \Longrightarrow v_2' = \frac{m_1}{m_2}(v_1 - v_1') + v_2'$$

As another application let us look at the following example. Suppose a man with mass 60 kg is at rest in outer space and holds a ball of mass 1 kg. He throws the ball with velocity 6 m/s to the right, what will man's velocity be?

In outer space there are no external forces for the system man + ball. So the total momentum is conserved. Since initially the man and the ball were at rest, total momentum was zero and it will continue to be zero. Therefore, after the man threw the ball, the ball has some momentum and man should have an equal and opposite momentum in order for the total momentum to vanish. So, man's velocity should be to the left and its magnitude must satisfy

$$60\ kg \cdot v_{man} = 1\ kg \cdot 6\ m/s \Longrightarrow\ v_{man} = 0.1 m/s$$

Homework

1. A fox is chasing a small rabbit. The momentum of the fox is equal to the momentum of the rabbit. Will the fox catch the rabbit?

2. A 80 kg jogger runs with a constant acceleration of 0.2 m/s^2 for 10 seconds. How his momentum changed during this time?

3. A 10 kg ball moving at a speed of 10 m/s hits a 5 kg ball which was at rest before the collision. After the collision the smaller ball starts moving at a speed 10 m/s. Find the velocity of the heavy ball after the collision. Neglect friction.

***4.** An astronaut of mass 100 kg approaches a cosmic ship of mass 50000 kg by pulling a negligible mass cable attached to the ship. Initial distance between the astronaut and the ship is 100 m and they both are initially at rest. What distance will the astronaut and the ship have traveled by their meeting time? *Hint: How are velocities of the astronaut and the ship related at every moment of time?*