MOTION WITH ACCELERATION

OCTOBER 18, 2020

Theory recap

Recall from the previous class that accelerated motion occurs whenever velocity changes. If velocity changes at a constant rate, this means that acceleration is constant and velocity ($\vec{v}$) dependence on time (t) is very simple:

$$\vec{v}(t) = \vec{v}_0 + \vec{a}\ t$$

Here $\vec{v}_0$ is initial velocity and $\vec{a}$ is acceleration.

But what about distance? We would like to know how far we travel when accelerating constantly. Say, we are planning to build a new airport on Long Island and one of the first things to decide is how much land do we need. What eats up most land in an airport? The answer is runway. Runway is where our airplanes will accelerate till they get enough speed to take off. We don't want it to be too short - no airplane will be able to take off then. We don't want it to be too long - there will be a lot of unused land, which is very expensive on Long Island, so our investors will not be happy. So, we need to know quite precisely, how far will the plane travel at constant acceleration before it reaches speed enough to take off. Moving with constant acceleration it will reach this speed at a certain time. So, **our task is to find how displacement l changes with time t for the airplane traveling with constant acceleration a.** Keeping our example with the airplane in mind, we will only consider motion along a straight line.

If speed was constant, we would know the answer right away: speed multiplied by time. The problem is that speed is changing. But motion with constant acceleration has a nice feature: speed changes at a constant rate. This leads to the fact that average velocity is an algebraic average of initial velocity and final velocity. For an airplane initially at rest (speed equals 0) after time t speed will be at. Algebraic average of initial and final velocity is then

$$v_{avg} = \frac{0 + at}{2} = \frac{at}{2}$$

Distance traveled is equal to average velocity multiplied by time:

$$l = v_{avg}t = \frac{at}{2}t = \frac{at^2}{2}$$

Remember that this formula applies for motion with constant acceleration starting at zero velocity.

Let us check dimensions. Unit of length is m, unit of acceleration is $\frac{m}{s^2}$ and unit of time is s. $m = \frac{m}{s^2}s^2$, so our formula passes the dimension check.

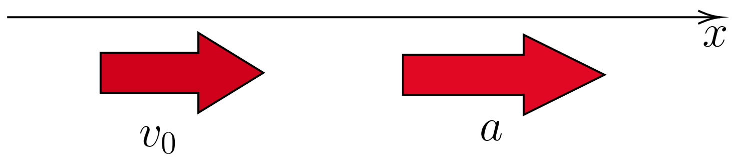
Now what if we started with some non-zero initial velocity v_0? Let us use the same idea of calculating average velocity: after time t velocity will be $v_0 + at$, so average velocity is

$$v_{avg} = \frac{v_0 + v_0 + at}{2} = v_0 + \frac{at}{2}$$

Then displacement is

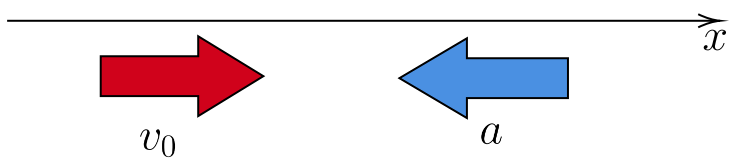
$$l = v_{avg}t = v_0 t + \frac{at}{2}t = v_0 t + \frac{at^2}{2}$$

Remember to pay attention to the signs. In the example of accelerating airplane acceleration a is directed in the same way as initial velocity v_0 (as shown on the picture below), so they both come with plus signs in the above equation.



As another example consider the problem of finding braking distance of a car. Suppose the car was moving with speed v_0 when the driver noticed an obstacle on a road and hit the brakes. Braking is happening at constant acceleration a. How far will the car move before stopping?

First we need to find what time is required to stop. Initial velocity is v_0 and final velocity is zero. Acceleration is directed oppositely to initial velocity (as shown on the picture below) because the driver is braking. Therefore acceleration should come with a minus sign in equations for final velocity v_f and braking distance l.



Knowing the relation between final velocity, initial velocity and acceleration we could find time:

$$v_f = v_0 - at = 0 \implies t = \frac{v_0}{a}$$

Now we could find braking distance knowing initial velocity, acceleration and time of the motion:

$$l = v_0 t - \frac{at^2}{2} = v_0\frac{v_0}{a} - \frac{a\left(\frac{v_0}{a}\right)^2}{2} = \frac{v_0^2}{2a}$$

Braking distance grows really fast with speed: if speed is increased by 2 times, braking distance becomes 4 times larger.

Homework

1. Let us finally find how long the runway in an airport should be! An airplane initially at rest accelerates with constant acceleration $a = 2\ m/s^2$ until it gets to the takeoff speed of $v_t = 80\ m/s$.
  (a) For how long does the airplane accelerate before taking off?
  (b) How far does the airplane move during this acceleration?
  (c) What is average velocity of the airplane during acceleration?

2. You have a bet with your friend that you could throw a ball higher than the roof of your school. Your school is 6 m high. You throw the ball vertically up with initial speed 15 m/s.

(a) In what time will it reach the highest point? (hint: at highest point it has to stop - if it has not stopped yet it would go even higher)

(b) What height will the ball reach? Did you win the bet?

(c) In what time after you threw it will it return to the ground?

***3.** You stay next to the front door of the first carriage of a train on a train station. The train starts to move with a constant acceleration. You notice that exactly in 3 seconds after the train started moving the front door of the second carriage passes you. How many carriages will move past you in the following 3 seconds? And then in the next 3 seconds? Assume that it's a very long train and all the carriages have the same length.